Answer on Question #78656 – Math – Analytic Geometry

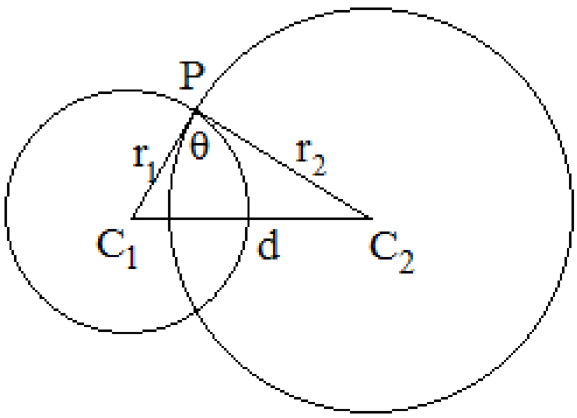
Question

Under what conditions on α do the spheres $x^2+y^2+z^2+\alpha x-y=0$ and $x^2+y^2+z^2+x+2z+1=0$ intersect each other at an angle of $45°$?

Solution

The angle of intersection of two spheres is the angle between their tangent planes at a common point.

Since the radii of the spheres through a common point are perpendicular to the tangent planes at the point, so the angle between the radii of the sphere at the common point is equal to the angle between their tangent planes, that is, the angle of intersection of the spheres



$$\theta = \cos^{-1}\left(\frac{r_1^2+r_2^2-d^2}{2r_1r_2}\right)$$

$$x^2+y^2+z^2+\alpha x-y=0$$

$$x^2+2\left(\frac{\alpha}{2}\right)x+\left(\frac{\alpha}{2}\right)^2+y^2-2\left(\frac{1}{2}\right)y+\left(\frac{1}{2}\right)^2+z^2-\left(\frac{\alpha}{2}\right)^2-\left(\frac{1}{2}\right)^2=0$$

$$\left(x+\frac{\alpha}{2}\right)^2+\left(y-\frac{1}{2}\right)^2+z^2=\frac{\alpha^2+1}{4}$$

$$x^2+y^2+z^2+x+2z+1=0$$

$$x^2+2\left(\frac{1}{2}\right)x+\left(\frac{1}{2}\right)^2+y^2+z^2+2z+1-\left(\frac{1}{2}\right)^2=0$$

$$\left(x+\frac{1}{2}\right)^2+y^2+(z+1)^2=\frac{1}{4}$$

$$C_1=\left(-\frac{\alpha}{2},\frac{1}{2},0\right), r_1=\frac{\sqrt{\alpha^2+1}}{2}$$

$$C_2=\left(-\frac{1}{2},0,-1\right), r_2=\frac{1}{2}$$

$$d^2=\left(-\frac{\alpha}{2}-\left(-\frac{1}{2}\right)\right)^2+\left(\frac{1}{2}-0\right)^2+(0-(-1))^2$$

$$d^2=\left(-\frac{\alpha}{2}+\frac{1}{2}\right)^2+\frac{5}{4}$$

$\theta = 45° => \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2} = \cos 45°$

$\frac{r_1^2 + r_2^2 - d^2}{2r_1r_2} = \frac{\sqrt{2}}{2}$

$\frac{\alpha^2 + 1}{4} + \frac{1}{4} - \left(-\frac{\alpha}{2} + \frac{1}{2}\right)^2 - \frac{5}{4} = \frac{\sqrt{\alpha^2 + 1}}{2} \cdot \frac{1}{2} \cdot \sqrt{2}$

$\alpha^2 + 1 + 1 - \alpha^2 + 2\alpha - 1 - 5 = \sqrt{2} \cdot \sqrt{\alpha^2 + 1}$

$2\alpha - 4 = \sqrt{2} \cdot \sqrt{\alpha^2 + 1}$

$4\alpha^2 - 16\alpha + 16 = 2\alpha^2 + 2$

$\alpha^2 - 8\alpha + 7 = 0$

$(\alpha - 1)(\alpha - 7) = 0$

$\alpha = 1$ or $\alpha = 7$

Answer: $\alpha = 1$ or $\alpha = 7$.