Answer on Question #78655 – Math – Analytic Geometry

Question

Find the equations of those tangent planes to the sphere $x^2 + y^2 + z^2 + 2x - 4y + 6z - 7 = 0$ which intersect in the line $6x - 3y - 23 = 0,\ 3z + 2 = 0$.

Solution

Let the tangent line be $Ax + By + Cz + D = 0$. Two easy points on the required line are $\left(0, -\frac{23}{3}, -\frac{2}{3}\right), \left(\frac{23}{6}, 0, -\frac{2}{3}\right)$.

For these to lie in the plane $Ax + By + Cz + D = 0$ we require

$$\begin{cases} 23A - 4C + 6D = 0 \\ -46B - 4C + 6D = 0 \end{cases}$$

Subtracting gives us $A = -2B$, and we also have $3D = 23B + 2C$.

The equation of the sphere can be rewritten in the following way

$$(x+1)^2 + (y-2)^2 + (z+3)^2 = 21,$$

so it has centre $(-1, 2, -3)$ and radius $\sqrt{21}$.

The centre must be a distance $\sqrt{21}$ from the tangent plane. Using the equation for the distance from a point to a plane we have

$$\frac{|-A + 2B - 3C + D|}{\sqrt{A^2 + B^2 + C^2}} = \sqrt{21}$$

Substituting the earlier relations for A and D we get

$$\left(4B - 3C + \frac{23}{3}B + \frac{2}{3}C\right)^2 = 21(5B^2 + C^2)$$

$$49\left(\frac{5}{3}B - \frac{1}{3}C\right)^2 = 21(5B^2 + C^2)$$

$$(4B + C)(B - 2C) = 0$$

So

$$\left[\begin{array}{l} C = \frac{1}{2}B \\ C = -4B \end{array}\right.$$

Therefore, one plane is $B = 2, C = 1, A = -4, D = 16$ or $4x - 2y - z = 16$ and the other plane is $B = 1, C = -4, A = -2, D = 5$ or $2x - y + 4z = 5$.

Answer: $4x - 2y - z = 16$ and $2x - y + 4z = 5$.