Answer on Question #78592 – Math – Analytic Geometry

Question

Check whether or not the conicoid represented by

$$5x^2 + 4y^2 - 4yz + 2xz + 2x - 4y - 8z + 2 = 0$$

is central or not. If it is, transform the equation by shifting the origin to the centre. Else, change any one coefficient to make the equation that of a central conicoid.

Solution

Theorem 1

The origin $O(0,0,0)$ is a centre of the conicoid

$$F(x,y,z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$$

if and only if $u = v = w = 0$

We have that

$$u = 1 \neq 0, v = -2 \neq 0, w = -4 \neq 0$$

Therefore, the origin $O(0,0,0)$ is not the centre of the conicoid.

Theorem 2

A conicoid S, given by the equation

$$F(x,y,z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$$

has the point $P(x_0, y_0, z_0)$ as a centre if and only if

$$\begin{cases} ax_0 + hy_0 + gz_0 + u = 0 \\ hx_0 + by_0 + fz_0 + v = 0 \\ gx_0 + fy_0 + cz_0 + w = 0 \end{cases}$$

We have that

$$\begin{cases} 5x_0 + z_0 + 1 = 0 \\ 4y_0 - 2z_0 - 2 = 0 \\ x_0 - 2y_0 - 4 = 0 \end{cases}$$

Augmented matrix

$$\begin{pmatrix} 5 & 0 & 1 & -1 \\ 0 & 4 & -2 & 2 \\ 1 & -2 & 0 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 0 & 1 & -1 \\ 0 & 4 & -2 & 2 \\ 1 & -2 & 0 & 4 \end{pmatrix} \xrightarrow{R_1/5} \begin{pmatrix} 1 & 0 & 1/5 & -1/5 \\ 0 & 4 & -2 & 2 \\ 1 & -2 & 0 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1/5 & -1/5 \\ 0 & 4 & -2 & 2 \\ 1 & -2 & 0 & 4 \end{pmatrix} \xrightarrow{R_3 - R_1} \begin{pmatrix} 1 & 0 & 1/5 & -1/5 \\ 0 & 4 & -2 & 2 \\ 0 & -2 & -1/5 & 21/5 \end{pmatrix}$$

$$\begin{pmatrix}1 & 0 & 1/5 & -1/5\\0 & 4 & -2 & 2\\0 & -2 & -1/5 & 21/5\end{pmatrix}\xrightarrow{R_2/4}\begin{pmatrix}1 & 0 & 1/5 & -1/5\\0 & 1 & -1/2 & 1/2\\0 & -2 & -1/5 & 21/5\end{pmatrix}$$

$$\begin{pmatrix}1 & 0 & 1/5 & -1/5\\0 & 1 & -1/2 & 1/2\\0 & -2 & -1/5 & 21/5\end{pmatrix}\xrightarrow{R_3+(2)R_2}\begin{pmatrix}1 & 0 & 1/5 & -1/5\\0 & 1 & -1/2 & 1/2\\0 & 0 & -6/5 & 26/5\end{pmatrix}$$

$$\begin{pmatrix}1 & 0 & 1/5 & -1/5\\0 & 1 & -1/2 & 1/2\\0 & 0 & -6/5 & 26/5\end{pmatrix}\xrightarrow{\left(-\frac{5}{6}\right)R_3}\begin{pmatrix}1 & 0 & 1/5 & -1/5\\0 & 1 & -1/2 & 1/2\\0 & 0 & 1 & -13/3\end{pmatrix}$$

$$\begin{pmatrix}1 & 0 & 1/5 & -1/5\\0 & 1 & -1/2 & 1/2\\0 & 0 & 1 & -13/3\end{pmatrix}\xrightarrow{R_1-\left(\frac{1}{5}\right)R_3}\begin{pmatrix}1 & 0 & 0 & 2/3\\0 & 1 & -1/2 & 1/2\\0 & 0 & 1 & -13/3\end{pmatrix}$$

$$\begin{pmatrix}1 & 0 & 0 & 2/3\\0 & 1 & -1/2 & 1/2\\0 & 0 & 1 & -13/3\end{pmatrix}\xrightarrow{R_2+\left(\frac{1}{2}\right)R_3}\begin{pmatrix}1 & 0 & 0 & 2/3\\0 & 1 & 0 & -5/3\\0 & 0 & 1 & -13/3\end{pmatrix}$$

$$\begin{cases} x_0=\dfrac{2}{3}\\ y_0=-\dfrac{5}{3}\\ z_0=-\dfrac{13}{3}\end{cases}$$

Then the point $P\left(\frac{2}{3},-\frac{5}{3},-\frac{13}{3}\right)$ is the centre of the coincoid.

In order for the conicoid to become central, it is necessary to apply a shift of the form

$$\begin{cases} x^*=x-\dfrac{2}{3}\\ y^*=y+\dfrac{5}{3}\\ z^*=z+\dfrac{13}{3}\end{cases}$$

Then

$$5x^2+4y^2-4yz+2xz+2x-4y-8z+2=0$$

$$5\left(x^*+\frac{2}{3}\right)^2+4\left(y^*-\frac{5}{3}\right)^2-4\left(y^*-\frac{5}{3}\right)\left(z^*-\frac{13}{3}\right)+2\left(x^*+\frac{2}{3}\right)\left(z^*-\frac{13}{3}\right)+$$
$$+2\left(x^*+\frac{2}{3}\right)-4\left(y^*-\frac{5}{3}\right)-8\left(z^*-\frac{13}{3}\right)+2=0$$

$$5x^{*2}+\frac{20}{3}x^*+\frac{20}{9}+4y^{*2}-\frac{40}{3}y^*+\frac{100}{9}-4y^*z^*+\frac{52}{3}y^*+\frac{20}{3}z^*-\frac{260}{9}+$$

$$+2x^*z^* - \frac{26}{3}x^* + \frac{4}{3}z^* - \frac{52}{9} + 2x^* + \frac{4}{3} - 4y^* + \frac{20}{3} - 8z^* + \frac{104}{3} + 2 = 0$$

$$5x^{*2} + 4y^{*2} - 4y^*z^* + 2x^*z^* + 0 \cdot x^* - 0 \cdot y^* + 0 \cdot z^* + \frac{70}{3} = 0$$

Hence

$$5x^2 + 4y^2 - 4yz + 2xz + 2x - 4y - 8z + 2 = 0 \ \rightarrow$$
$$\rightarrow 5x^{*2} + 4y^{*2} - 4y^*z^* + 2x^*z^* + \frac{70}{3} = 0$$

Answer:

1) The conicoid represented by

$$5x^2 + 4y^2 - 4yz + 2xz + 2x - 4y - 8z + 2 = 0$$

is not central.

2) The point $P\left(\frac{2}{3}, -\frac{5}{3}, -\frac{13}{3}\right)$ is the centre of the coincoid.

3) It is necessary to apply a shift of the form

$$\begin{cases} x^* = x - \frac{2}{3} \\ y^* = y + \frac{5}{3} \\ z^* = z + \frac{13}{3} \end{cases}$$

to make the conicoid central.

$$5x^2 + 4y^2 - 4yz + 2xz + 2x - 4y - 8z + 2 = 0 \ \rightarrow$$
$$\rightarrow 5x^{*2} + 4y^{*2} - 4y^*z^* + 2x^*z^* + \frac{70}{3} = 0$$