ANSWER on Question #78511 – Math – Differential Equations

QUESTION

Solve

$$\frac{\partial^2 U}{\partial x^2} = 9 \cdot \frac{\partial^2 U}{\partial x^2}$$

1. $U(x, y) = \cos(3x - y)$
2. $U(x, y) = x^2 + y^2$
3. $U(x, y) = \sin(3x - y)$
4. $U(x, y) = e^{-3/pix} \cdot \sin(\pi y)$

SOLUTION

Remarks related to the condition of the problem:

1. We see that the answers are given as functions of two variables - $U(x, y)$. Therefore, the given equation MUST look like this

$$1.\quad \frac{\partial^2 U}{\partial x^2} = 9 \cdot \frac{\partial^2 U}{\partial y^2} \quad or \quad 2.\quad \frac{\partial^2 U}{\partial y^2} = 9 \cdot \frac{\partial^2 U}{\partial x^2}$$

For each of these cases, we check the solutions provided.

2. Since there are no boundary conditions, we can not apply the method of separation of variables and reduce this problem to the Sturm-Liouville problem.

3. Variant # 4 for the formula editor looks like

$$U(x, y) = e^{-3/pix} \cdot \sin(\pi y) \to U(x, y) = e^{-\frac{3}{\pi x}} \cdot \sin(\pi y)$$

But I think that should be so

$$U(x, y) = e^{-3/pix} \cdot \sin(\pi y) \to U(x, y) = e^{-3\pi x} \cdot \sin(\pi y)$$

Therefore, two cases will also be considered.

We will use this method of solution: we calculate all the partial derivatives and substitute them in the original equation. If equality holds, then it is a solution for us.

1 CASE:

$$\frac{\partial^2 U}{\partial x^2} = 9 \cdot \frac{\partial^2 U}{\partial y^2}$$

1. $U(x, y) = \cos(3x - y)$

$$U(x, y) = \cos(3x - y) \rightarrow \begin{cases} \dfrac{\partial U}{\partial x} = -3\sin(3x - y) \\ \dfrac{\partial^2 U}{\partial x^2} = -9\cos(3x - y) \\ \dfrac{\partial U}{\partial y} = \sin(3x - y) \\ \dfrac{\partial^2 U}{\partial y^2} = -\cos(3x - y) \end{cases}$$

Then,

$$\begin{cases} \dfrac{\partial^2 U}{\partial x^2} = -9\cos(3x - y) \\ 9 \cdot \dfrac{\partial^2 U}{\partial y^2} = 9 \cdot [-\cos(3x - y)] = -9\cos(3x - y) \end{cases} \rightarrow \frac{\partial^2 U}{\partial x^2} = 9 \cdot \frac{\partial^2 U}{\partial y^2}$$

Conclusion,

$$\boxed{U(x, y) = \cos(3x - y) - is\ solution\ of\ the\ given\ equation}$$

2. $U(x,y) = x^2 + y^2$

$$U(x,y) = x^2 + y^2 \rightarrow \begin{cases} \dfrac{\partial U}{\partial x} = 2x \\ \dfrac{\partial^2 U}{\partial x^2} = 2 \\ \dfrac{\partial U}{\partial y} = 2y \\ \dfrac{\partial^2 U}{\partial y^2} = 2 \end{cases}$$

Then,

$$\begin{cases} \dfrac{\partial^2 U}{\partial x^2} = 2 \\ 9 \cdot \dfrac{\partial^2 U}{\partial y^2} = 9 \cdot 2 = 18 \end{cases} \rightarrow \frac{\partial^2 U}{\partial x^2} = 2 \neq 18 = 9 \cdot \frac{\partial^2 U}{\partial y^2}$$

Conclusion,

$$\boxed{U(x,y) = x^2 + y^2 - is\ not\ solution\ of\ the\ given\ equation}$$

3. $U(x,y) = \sin(3x - y)$

$$U(x,y) = \sin(3x - y) \rightarrow \begin{cases} \dfrac{\partial U}{\partial x} = 3\cos(3x - y) \\ \dfrac{\partial^2 U}{\partial x^2} = -9\sin(3x - y) \\ \dfrac{\partial U}{\partial y} = -\cos(3x - y) \\ \dfrac{\partial^2 U}{\partial y^2} = -\sin(3x - y) \end{cases}$$

Then,

$$\begin{cases} \dfrac{\partial^2 U}{\partial x^2} = -9\sin(3x - y) \\ 9 \cdot \dfrac{\partial^2 U}{\partial y^2} = 9 \cdot [-\sin(3x - y)] = -9\sin(3x - y) \end{cases} \rightarrow \frac{\partial^2 U}{\partial x^2} = 9 \cdot \frac{\partial^2 U}{\partial y^2}$$

Conclusion,

$$\boxed{U(x,y)=\sin(3x-y) - \textit{is solution of the given equation}}$$

4.A. $U(x,y)=e^{-3\pi x}\cdot\sin(\pi y)$

$$U(x,y)=e^{-3\pi x}\cdot\sin(\pi y)\rightarrow\begin{cases}\dfrac{\partial U}{\partial x}=-3\pi\cdot e^{-3\pi x}\cdot\sin(\pi y)\\ \dfrac{\partial^2 U}{\partial x^2}=9\pi^2\cdot e^{-3\pi x}\cdot\sin(\pi y)\\ \dfrac{\partial U}{\partial y}=\pi\cdot e^{-3\pi x}\cdot\cos(\pi y)\\ \dfrac{\partial^2 U}{\partial y^2}=-\pi^2\cdot e^{-3\pi x}\cdot\sin(\pi y)\end{cases}$$

Then,

$$\begin{cases}\dfrac{\partial^2 U}{\partial x^2}=9\pi^2\cdot e^{-3\pi x}\cdot\sin(\pi y)\\ 9\cdot\dfrac{\partial^2 U}{\partial y^2}=9\cdot[-\pi^2\cdot e^{-3\pi x}\cdot\sin(\pi y)]=-9\pi^2\cdot e^{-3\pi x}\cdot\sin(\pi y)\end{cases}\rightarrow$$

$$\frac{\partial^2 U}{\partial x^2}=9\pi^2\cdot e^{-3\pi x}\cdot\sin(\pi y)\neq-9\pi^2\cdot e^{-3\pi x}\cdot\sin(\pi y)=9\cdot\frac{\partial^2 U}{\partial y^2}$$

Conclusion,

$$\boxed{U(x,y)=e^{-3\pi x}\cdot\sin(\pi y) - \textit{is not solution of the given equation}}$$

4.B. $U(x,y) = e^{-\frac{3}{\pi x}} \cdot \sin(\pi y)$

$$U(x,y) = e^{-\frac{3}{\pi x}} \cdot \sin(\pi y) \rightarrow \begin{cases} \dfrac{\partial U}{\partial x} = \dfrac{3}{\pi x^2} \cdot e^{-\frac{3}{\pi x}} \cdot \sin(\pi y) \\ \dfrac{\partial^2 U}{\partial x^2} = -\dfrac{6}{\pi x^3} \cdot e^{-\frac{3}{\pi x}} \cdot \sin(\pi y) + \dfrac{9}{\pi^2 \cdot x^4} \cdot e^{-\frac{3}{\pi x}} \cdot \sin(\pi y) \\ \dfrac{\partial U}{\partial y} = \pi \cdot e^{-\frac{3}{\pi x}} \cdot \cos(\pi y) \\ \dfrac{\partial^2 U}{\partial y^2} = -\pi^2 \cdot e^{-\frac{3}{\pi x}} \cdot \sin(\pi y) \end{cases}$$

Then,

$$\begin{cases} \dfrac{\partial^2 U}{\partial x^2} = \left(\dfrac{9}{\pi^2 x^4} - \dfrac{6}{\pi x^3}\right) \cdot e^{-\frac{3}{\pi x}} \cdot \sin(\pi y) \\ 9 \cdot \dfrac{\partial^2 U}{\partial y^2} = 9 \cdot \left[-\pi^2 \cdot e^{-\frac{3}{\pi x}} \cdot \sin(\pi y)\right] = -9\pi^2 \cdot e^{-\frac{3}{\pi x}} \cdot \sin(\pi y) \end{cases} \rightarrow$$

$$\frac{\partial^2 U}{\partial x^2} = \left(\frac{9}{\pi^2 x^4} - \frac{6}{\pi x^3}\right) \cdot e^{-\frac{3}{\pi x}} \cdot \sin(\pi y) \neq -9\pi^2 \cdot e^{-\frac{3}{\pi x}} \cdot \sin(\pi y) = 9 \cdot \frac{\partial^2 U}{\partial y^2}$$

Conclusion,

$$\boxed{U(x,y) = e^{-\frac{3}{\pi x}} \cdot \sin(\pi y) - \textit{is not solution of the given equation}}$$

2 CASE:

$$\frac{\partial^2 U}{\partial y^2} = 9 \cdot \frac{\partial^2 U}{\partial x^2}$$

1. $U(x, y) = \cos(3x - y)$

$$U(x, y) = \cos(3x - y) \rightarrow \begin{cases} \dfrac{\partial U}{\partial x} = -3\sin(3x - y) \\ \dfrac{\partial^2 U}{\partial x^2} = -9\cos(3x - y) \\ \dfrac{\partial U}{\partial y} = \sin(3x - y) \\ \dfrac{\partial^2 U}{\partial y^2} = -\cos(3x - y) \end{cases}$$

Then,

$$\begin{cases} \dfrac{\partial^2 U}{\partial y^2} = -\cos(3x - y) \\ 9 \cdot \dfrac{\partial^2 U}{\partial x^2} = 9 \cdot [-9\cos(3x - y)] = -81\cos(3x - y) \end{cases} \rightarrow \frac{\partial^2 U}{\partial y^2} \neq 9 \cdot \frac{\partial^2 U}{\partial x^2}$$

Conclusion,

$$\boxed{U(x, y) = \cos(3x - y) - is\ not\ solution\ of\ the\ given\ equation}$$

2. $U(x,y) = x^2 + y^2$

$$U(x,y) = x^2 + y^2 \to \begin{cases} \dfrac{\partial U}{\partial x} = 2x \\ \dfrac{\partial^2 U}{\partial x^2} = 2 \\ \dfrac{\partial U}{\partial y} = 2y \\ \dfrac{\partial^2 U}{\partial y^2} = 2 \end{cases}$$

Then,

$$\begin{cases} \dfrac{\partial^2 U}{\partial y^2} = 2 \\ 9 \cdot \dfrac{\partial^2 U}{\partial x^2} = 9 \cdot 2 = 18 \end{cases} \to \frac{\partial^2 U}{\partial y^2} = 2 \neq 18 = 9 \cdot \frac{\partial^2 U}{\partial x^2}$$

Conclusion,

$$\boxed{U(x,y) = x^2 + y^2 - is\ not\ solution\ of\ the\ given\ equation}$$

3. $U(x,y) = \sin(3x - y)$

$$U(x,y) = \sin(3x - y) \to \begin{cases} \dfrac{\partial U}{\partial x} = 3\cos(3x - y) \\ \dfrac{\partial^2 U}{\partial x^2} = -9\sin(3x - y) \\ \dfrac{\partial U}{\partial y} = -\cos(3x - y) \\ \dfrac{\partial^2 U}{\partial y^2} = -\sin(3x - y) \end{cases}$$

Then,

$$\begin{cases} \dfrac{\partial^2 U}{\partial y^2} = -\sin(3x - y) \\ 9 \cdot \dfrac{\partial^2 U}{\partial x^2} = 9 \cdot [-9\sin(3x - y)] = -81\sin(3x - y) \end{cases} \to \frac{\partial^2 U}{\partial y^2} \neq 9 \cdot \frac{\partial^2 U}{\partial x^2}$$

Conclusion,

$$\boxed{U(x,y) = \sin(3x - y) - \textit{is not solution of the given equation}}$$

4.A. $U(x,y) = e^{-3\pi x} \cdot \sin(\pi y)$

$$U(x,y) = e^{-3\pi x} \cdot \sin(\pi y) \rightarrow \begin{cases} \dfrac{\partial U}{\partial x} = -3\pi \cdot e^{-3\pi x} \cdot \sin(\pi y) \\ \dfrac{\partial^2 U}{\partial x^2} = 9\pi^2 \cdot e^{-3\pi x} \cdot \sin(\pi y) \\ \dfrac{\partial U}{\partial y} = \pi \cdot e^{-3\pi x} \cdot \cos(\pi y) \\ \dfrac{\partial^2 U}{\partial y^2} = -\pi^2 \cdot e^{-3\pi x} \cdot \sin(\pi y) \end{cases}$$

Then,

$$\begin{cases} \dfrac{\partial^2 U}{\partial y^2} = -\pi^2 \cdot e^{-3\pi x} \cdot \sin(\pi y) \\ 9 \cdot \dfrac{\partial^2 U}{\partial x^2} = 9 \cdot [9\pi^2 \cdot e^{-3\pi x} \cdot \sin(\pi y)] = 81\pi^2 \cdot e^{-3\pi x} \cdot \sin(\pi y) \end{cases} \rightarrow$$

$$\frac{\partial^2 U}{\partial y^2} = -\pi^2 \cdot e^{-3\pi x} \cdot \sin(\pi y) \neq 81\pi^2 \cdot e^{-3\pi x} \cdot \sin(\pi y) = 9 \cdot \frac{\partial^2 U}{\partial x^2}$$

Conclusion,

$$\boxed{U(x,y) = e^{-3\pi x} \cdot \sin(\pi y) - \textit{is not solution of the given equation}}$$

4.B. $U(x,y) = e^{-\frac{3}{\pi x}} \cdot \sin(\pi y)$

$$U(x,y) = e^{-\frac{3}{\pi x}} \cdot \sin(\pi y) \rightarrow \begin{cases} \dfrac{\partial U}{\partial x} = \dfrac{3}{\pi x^2} \cdot e^{-\frac{3}{\pi x}} \cdot \sin(\pi y) \\ \dfrac{\partial^2 U}{\partial x^2} = -\dfrac{6}{\pi x^3} \cdot e^{-\frac{3}{\pi x}} \cdot \sin(\pi y) + \dfrac{9}{\pi^2 \cdot x^4} \cdot e^{-\frac{3}{\pi x}} \cdot \sin(\pi y) \\ \dfrac{\partial U}{\partial y} = \pi \cdot e^{-\frac{3}{\pi x}} \cdot \cos(\pi y) \\ \dfrac{\partial^2 U}{\partial y^2} = -\pi^2 \cdot e^{-\frac{3}{\pi x}} \cdot \sin(\pi y) \end{cases}$$

Then,

$$\begin{cases} \dfrac{\partial^2 U}{\partial y^2} = -\pi^2 \cdot e^{-\frac{3}{\pi x}} \cdot \sin(\pi y) \\ 9 \cdot \dfrac{\partial^2 U}{\partial x^2} = 9 \cdot \left[\left(\dfrac{9}{\pi^2 x^4} - \dfrac{6}{\pi x^3} \right) \cdot e^{-\frac{3}{\pi x}} \cdot \sin(\pi y) \right] = \left(\dfrac{81}{\pi^2 x^4} - \dfrac{54}{\pi x^3} \right) \cdot e^{-\frac{3}{\pi x}} \cdot \sin(\pi y) \end{cases} \rightarrow$$

$$\frac{\partial^2 U}{\partial y^2} = -\pi^2 \cdot e^{-\frac{3}{\pi x}} \cdot \sin(\pi y) \neq \left(\frac{81}{\pi^2 x^4} - \frac{54}{\pi x^3} \right) \cdot e^{-\frac{3}{\pi x}} \cdot \sin(\pi y) = 9 \cdot \frac{\partial^2 U}{\partial x^2}$$

Conclusion,

$$\boxed{U(x,y) = e^{-\frac{3}{\pi x}} \cdot \sin(\pi y) - is\ not\ solution\ of\ the\ given\ equation}$$

ANSWER

$$if \quad \frac{\partial^2 U}{\partial x^2} = 9 \cdot \frac{\partial^2 U}{\partial x^2} \rightarrow \frac{\partial^2 U}{\partial x^2} = 9 \cdot \frac{\partial^2 U}{\partial y^2}$$

Then, solutions are

$$1.\ \ U(x,y) = \cos(3x - y) \quad and \quad 3.\ \ U(x,y) = \sin(3x - y)$$

$$if \quad \frac{\partial^2 U}{\partial x^2} = 9 \cdot \frac{\partial^2 U}{\partial x^2} \rightarrow \frac{\partial^2 U}{\partial y^2} = 9 \cdot \frac{\partial^2 U}{\partial x^2}$$

Then, solutions are

$$1.\ \ U(x,y)=\cos(3x-y)\quad and\quad 3.\ \ U(x,y)=\sin(3x-y)$$

$$None\ of\ the\ presented\ variants\ is\ a\ solution\ of\ this\ equation$$