Answer on Question #78460 – Math – Algebra

Question

Let $x_1, \ldots x_n \in \mathbb{R}, n \geq 2,$ such that $0 < x_1 \leq x_2 \leq \ldots \leq x_n,$ and $\frac{1}{1+x_1} + \frac{1}{1+x_2} + \ldots + \frac{1}{1+x_n} = 1,$ then show that $\sqrt{x_1} + \sqrt{x_2} + \ldots + \sqrt{x_n} \geq (n-1)\left(\frac{1}{\sqrt{x_1}} + \ldots + \frac{1}{\sqrt{x_n}}\right).$

Solution

0) This problem originally appeared on the Vojtěch Jarník Competition, in 2002.

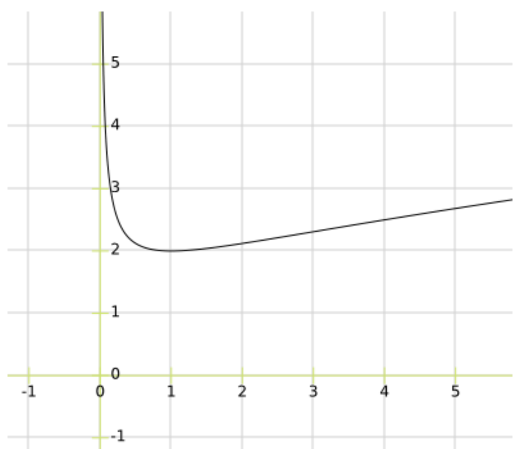
1) Consider a case when $n = 2$.

$1 = \frac{1}{1+x_1} + \frac{1}{1+x_2} = \frac{x_1+x_2+2}{x_1x_2+x_1+x_2+1} \Longrightarrow x_1x_2 = 2 - 1 = 1$

$\sqrt{x_1} + \sqrt{x_2} - \frac{1}{\sqrt{x_1}} - \frac{1}{\sqrt{x_2}} = \frac{(\sqrt{x_1}\sqrt{x_2}-1)\sqrt{x_1}+(\sqrt{x_1}\sqrt{x_2}-1)\sqrt{x_2}}{\sqrt{x_1}\sqrt{x_2}} = 0 \geq 0.$

2) First, assume that $x_1 \leq 1$ and $i \geq 2$, then $\frac{1}{1+x_i} \leq \frac{1}{1+x_2} + \ldots + \frac{1}{1+x_n} \leq 1 - \frac{1}{1+x_1} = \frac{x_1}{1+x_1} = \frac{1}{1+\frac{1}{x_1}},$ then $x_i \geq 1$, and $x_i \geq 1/x_1$.

Next, consider a sequence $a_i = \sqrt{x_i} + \frac{1}{\sqrt{x_i}} = \frac{x_i+1}{\sqrt{x_i}}$. It is obvious that $0 < 1 \leq a_1 \leq a_2 \leq \ldots \leq a_n$ if $1 < x_1 \leq x_2 \leq \ldots \leq x_n$ and $0 \leq a_1 \leq 1 \leq a_2 \leq \ldots \leq a_n$ if $x_1 \leq 1 \leq \frac{1}{x_1} \leq x_2 \leq \ldots \leq x_n$:



Now, consider a sequence $b_i = \frac{1}{1+x_i}$. It is obvious that $b_1 \geq b_2 \geq \ldots \geq b_n > 0$.

The Chebyshev's inequality gives that $\sum a_i * \sum b_i \equiv \sum\left(\sqrt{x_i} + \frac{1}{\sqrt{x_i}}\right) * 1 \geq n \sum a_i b_i \equiv n \sum \frac{1}{x_i}$. This completes the proof.

The equality holds if and only $a_i = a_1$ or $b_i = b_1$. It is easy to see that the latter means that $x_1 = x_i = n - 1$. The former yields one more case: $x_1 \leq 1 \leq \frac{1}{x_1} = x_i$. Since $1 = \frac{1}{1+\frac{1}{x_2}} + \frac{(n-1)}{1+x_2} = \frac{n-1+x_2}{1+x_2}$, it follows that $n - 1 = 1$.

Answer:

The statement can be proved using Chebyshev's inequality. The equality holds if and only if $x_1 = x_i = n - 1$ $(n \geq 2)$ or $x_1 < 1 < \frac{1}{x_1} = x_2$ $(n = 2)$.