Answer on Question #78459 – Math – Algebra

Question

Prove that

$$\frac{1}{2}(x+y+z) \le \frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y} \text{ for } x, y, z > 0$$

Solution

Use Cauchy-Buniakowsky-Schwarz inequality

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \ge (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2$$

with equality if and only if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$

Then

$$[(y+z)+(x+z)+(x+y)] \times \left[\frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y}\right] \ge$$

$$\ge \left(\sqrt{y+z} \cdot \frac{x}{\sqrt{y+z}} + \sqrt{x+z} \cdot \frac{y}{\sqrt{x+z}} + \sqrt{x+y} \cdot \frac{z}{\sqrt{x+y}}\right)^2$$

Divide each side by $2(x+y+z)$

$$\frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y} \ge \frac{x+y+z}{2}$$

Equality if and only if $\frac{x}{y+z} = \frac{y}{x+z} = \frac{z}{x+y}$