Question 1. *Suppose S is a nonempty open set that isn't the whole real line. Show that there is a sequence of elements of S that converges to an element of $C(S)$.*

Solution. Take $x \in S$ (we can do this, because S is nonempty). Since S is open, there is an open interval $(x - \varepsilon, x + \varepsilon) \subset S$ for some $\varepsilon > 0$. Set $\varepsilon_0 = \sup\{\varepsilon > 0 \mid (x - \varepsilon, x + \varepsilon) \subset S\}$. Note that $\varepsilon_0 < \infty$, since otherwise $S = \mathbb{R}$. Then either $x - \varepsilon_0$ or $x + \varepsilon_0$ does not belong to S. Indeed, otherwise we can find an open interval in S, which contains $(x - \varepsilon_0, x + \varepsilon_0)$ and thus ε_0 is not maximal. Let $x - \varepsilon_0 \notin S$ (the case $x + \varepsilon_0 \notin S$ is similar). Let $N \in \mathbb{N}$ be a positive integer such that $\frac{1}{N} < \varepsilon_0$. Then $\frac{1}{N+n} < \varepsilon_0$ and thus $x_n = x - \varepsilon_0 + \frac{1}{N+n} \in S$ for all $n \in \mathbb{N}$. Finally note that $\lim x_n = x - \varepsilon_0 \notin S$, i.e. $\lim x_n \in C(S)$. □