ANSWER on Question #78083 – Math – Calculus

QUESTION

Evaluate the following integral

$$\int_0^1 \frac{3x^2 - x^2 + 2x - 4}{\sqrt{x^2 - 3x + 2}} dx$$

SOLUTION

We recall some table integrals that we will need to solve this problem

$$1)\quad \int \sqrt{x^2 - a^2} dx = \frac{1}{2}\left(x\sqrt{x^2 - a^2} - a^2 \ln\left(x + \sqrt{x^2 - a^2}\right)\right) + Const$$

$$2)\quad \int \frac{dx}{\sqrt{x}} = 2\sqrt{x} + Const$$

$$3)\quad \int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left|\frac{x + \sqrt{x^2 - a^2}}{a}\right| + Const$$

$$4)\quad \int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a)$$

(More information: https://en.wikipedia.org/wiki/List_of_integrals_of_irrational_functions)

(More information: https://en.wikipedia.org/wiki/Fundamental_theorem_of_calculus)

In our case,

$$\int_0^1 \frac{3x^2 - x^2 + 2x - 4}{\sqrt{x^2 - 3x + 2}} dx$$

1 STEP: We transform this integral

$$\int_0^1 \frac{3x^2-x^2+2x-4}{\sqrt{x^2-3x+2}}dx=\int_0^1 \frac{2x^2+2x-4}{\sqrt{x^2-3x+2}}dx=\int_0^1 \frac{2\cdot(x^2+x-2)}{\sqrt{x^2-3x+2}}dx=2\cdot\int_0^1 \frac{x^2+x-2}{\sqrt{x^2-3x+2}}dx=$$

$$=2\cdot\int_0^1 \frac{x^2-3x+4x+2-4}{\sqrt{x^2-3x+2}}dx=2\cdot\int_0^1 \frac{(x^2-3x+2)+4x-4}{\sqrt{x^2-3x+2}}dx=$$

$$=2\cdot\int_0^1\left(\frac{(x^2-3x+2)}{\sqrt{x^2-3x+2}}+\frac{4x-4}{\sqrt{x^2-3x+2}}\right)dx=2\cdot\int_0^1\sqrt{x^2-3x+2}dx+2\cdot\int_0^1\frac{4x-4}{\sqrt{x^2-3x+2}}dx=$$

$$=2\cdot\int_0^1\sqrt{x^2-2\cdot x\cdot\frac{3}{2}+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2+2}dx+2\cdot\int_0^1\frac{2\cdot(2x-2)}{\sqrt{x^2-3x+2}}dx=$$

$$=2\cdot\int_0^1\sqrt{\left(x-\frac{3}{2}\right)^2-\frac{9}{4}+2}dx+2\cdot2\cdot\int_0^1\frac{2x-3+1}{\sqrt{x^2-3x+2}}dx=$$

$$=2\cdot\int_0^1\sqrt{\left(x-\frac{3}{2}\right)^2-\frac{1}{4}}dx+4\cdot\int_0^1\frac{2x-3}{\sqrt{x^2-3x+2}}dx+4\cdot\int_0^1\frac{1}{\sqrt{x^2-3x+2}}dx=I_1+I_2+I_3$$

2.A STEP: Calculate I_1

$$I_1=2\cdot\int_0^1\sqrt{\left(x-\frac{3}{2}\right)^2-\frac{1}{4}}dx=\begin{bmatrix}x-\frac{3}{2}=t\\ dx=dt\end{bmatrix}=2\cdot\int_a^b\sqrt{t^2-\left(\frac{1}{2}\right)^2}\,dx=[We\ use\ the\ first\ table\ integral]=$$

$$=2\cdot\frac{1}{2}\cdot\left(t\sqrt{t^2-\left(\frac{1}{2}\right)^2}-\left(\frac{1}{2}\right)^2\ln\left|t+\sqrt{t^2-\left(\frac{1}{2}\right)^2}\right|\right)\Bigg|_a^b=$$

$$=\left(x-\frac{3}{2}\right)\sqrt{\left(x-\frac{3}{2}\right)^2-\frac{1}{4}}-\frac{1}{4}\ln\left|x-\frac{3}{2}+\sqrt{\left(x-\frac{3}{2}\right)^2-\frac{1}{4}}\right|\Bigg|_0^1=$$

$$=\left[\left(1-\frac{3}{2}\right)\sqrt{\left(1-\frac{3}{2}\right)^2-\frac{1}{4}}-\frac{1}{4}\ln\left|1-\frac{3}{2}+\sqrt{\left(1-\frac{3}{2}\right)^2-\frac{1}{4}}\right|\right]-$$

$$-\left[\left(0-\frac{3}{2}\right)\sqrt{\left(0-\frac{3}{2}\right)^2-\frac{1}{4}}-\frac{1}{4}\ln\left|0-\frac{3}{2}+\sqrt{\left(0-\frac{3}{2}\right)^2-\frac{1}{4}}\right|\right]=$$

$$=\left[-\frac{1}{2}\cdot\sqrt{\left(\frac{1}{2}\right)^2-\frac{1}{4}}-\frac{1}{4}\ln\left|-\frac{1}{2}+\sqrt{\left(\frac{1}{2}\right)^2-\frac{1}{4}}\right|\right]-\left[-\frac{3}{2}\cdot\sqrt{\left(\frac{3}{2}\right)^2-\frac{1}{4}}-\frac{1}{4}\ln\left|-\frac{3}{2}+\sqrt{\left(\frac{3}{2}\right)^2-\frac{1}{4}}\right|\right]=$$

$$=\left[-\frac{1}{2}\cdot 0-\frac{1}{4}\ln\left|-\frac{1}{2}-\frac{1}{4}\cdot 0\right|\right]-\left[-\frac{3}{2}\cdot\sqrt{\frac{9}{4}-\frac{1}{4}}-\frac{1}{4}\ln\left|-\frac{3}{2}+\sqrt{\frac{9}{4}-\frac{1}{4}}\right|\right]=$$

$$=-\frac{1}{4}\cdot\ln\left|-\frac{1}{2}\right|-\left[-\frac{3}{2}\cdot\sqrt{2}-\frac{1}{4}\cdot\ln\left|-\frac{3}{2}+\sqrt{2}\right|\right]=-\frac{1}{4}\cdot\ln\left(\frac{1}{2}\right)-\left[-\frac{3\sqrt{2}}{2}-\frac{1}{4}\cdot\ln\left|-\frac{3}{2}+\frac{2\sqrt{2}}{2}\right|\right]=$$

$$=-\frac{1}{4}\cdot\ln\frac{1}{2}+\frac{3\sqrt{2}}{2}+\frac{1}{4}\ln\left|\frac{-3+2\sqrt{2}}{2}\right|=\frac{3\sqrt{2}}{2}+\frac{1}{4}\cdot\ln\frac{\frac{3-2\sqrt{2}}{2}}{\frac{1}{2}}=\frac{3\sqrt{2}}{2}+\frac{1}{4}\cdot\ln\left(3-2\sqrt{2}\right)=$$

$$=\frac{1}{4}\cdot\left(6\sqrt{2}+\ln\left(3-2\sqrt{2}\right)\right)\approx 1.680634$$

$$\boxed{I_1=\frac{1}{4}\cdot\left(6\sqrt{2}+\ln\left(3-2\sqrt{2}\right)\right)\approx 1.680634}$$

1.B STEP: Calculate I_2

$$I_2=4\cdot\int_0^1\frac{2x-3}{\sqrt{x^2-3x+2}}dx=\begin{bmatrix}x^2-3x+2=t\\(2x-3)dx=dt\end{bmatrix}=4\cdot\int_a^b\frac{dt}{\sqrt{t}}=[We\ use\ the\ second\ table\ integral]=$$

$$=4\cdot 2\cdot\sqrt{t}\Big|_a^b=8\cdot\sqrt{x^2-3x+2}\Big|_0^1=\left(8\cdot\sqrt{1^2-3\cdot 1+2}\right)-\left(8\cdot\sqrt{0^2-3\cdot 0+2}\right)=$$

$$=\left(8\cdot\sqrt{1-3+2}\right)-\left(8\cdot\sqrt{0-0+2}\right)=-8\sqrt{2}\approx -11.313708$$

$$\boxed{I_2=-8\sqrt{2}\approx -11.313708}$$

1.C STEP: Calculate I_3

$$I_3=4\cdot\int_0^1\frac{1}{\sqrt{x^2-3x+2}}dx=4\cdot\int_0^1\frac{1}{\sqrt{x^2-2\cdot x\cdot\frac{3}{2}+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2+2}}dx=$$

$$=4\cdot\int_0^1\frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\frac{9}{4}+2}}dx=4\cdot\int_0^1\frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\frac{1}{4}}}dx=\left[\begin{matrix}x-\frac{3}{2}=t\\ dx=dt\end{matrix}\right]=$$

$$=4\cdot\int_a^b\frac{1}{\sqrt{t^2-\left(\frac{1}{2}\right)^2}}dt=[We\ use\ the\ third\ table\ integral]=4\cdot\ln\left|\frac{t+\sqrt{t^2-\left(\frac{1}{2}\right)^2}}{\frac{1}{2}}\right|\Bigg|_a^b=$$

$$=4\cdot\ln\left|2\cdot\left(x-\frac{3}{2}\right)+2\cdot\sqrt{\left(x-\frac{3}{2}\right)^2-\frac{1}{4}}\right|\Bigg|_0^1=$$

$$=\left[4\cdot\ln\left|2\cdot\left(1-\frac{3}{2}\right)+2\cdot\sqrt{\left(1-\frac{3}{2}\right)^2-\frac{1}{4}}\right|\right]-\left[4\cdot\ln\left|2\cdot\left(0-\frac{3}{2}\right)+2\cdot\sqrt{\left(0-\frac{3}{2}\right)^2-\frac{1}{4}}\right|\right]=$$

$$=\left[4\cdot\ln\left|2\cdot\left(-\frac{1}{2}\right)+2\cdot\sqrt{\frac{1}{4}-\frac{1}{4}}\right|\right]-\left[4\cdot\ln\left|-3+2\cdot\sqrt{\frac{9}{4}-\frac{1}{4}}\right|\right]=$$

$$=[4\cdot\ln|-1+2\cdot 0|]-\left[4\cdot\ln\left|-3+2\cdot\sqrt{2}\right|\right]=4\cdot\ln 1-4\cdot\ln\left|2\sqrt{2}-3\right|=-4\cdot\ln\left(3-2\sqrt{2}\right)\approx 7.050989$$

$$\boxed{I_3=-4\cdot\ln\left(3-2\sqrt{2}\right)\approx 7.050989}$$

General Conclusion,

$$\int_0^1\frac{3x^2-x^2+2x-4}{\sqrt{x^2-3x+2}}dx=I_1+I_2+I_3=\frac{1}{4}\cdot\left(6\sqrt{2}+\ln\left(3-2\sqrt{2}\right)\right)-8\sqrt{2}-4\cdot\ln\left(3-2\sqrt{2}\right)=$$

$$=\frac{3\sqrt{2}}{2}+\frac{1}{4}\cdot\ln\left(3-2\sqrt{2}\right)-8\sqrt{2}-4\cdot\ln\left(3-2\sqrt{2}\right)=\left(\frac{3}{2}-\frac{16}{2}\right)\cdot\sqrt{2}+\left(\frac{1}{4}-\frac{16}{4}\right)\cdot\ln\left(3-2\sqrt{2}\right)=$$

$$=-\frac{13\sqrt{2}}{2}-\frac{15}{4}\cdot\ln\left(3-2\sqrt{2}\right)=-\frac{1}{4}\cdot\left(26\sqrt{2}+15\cdot\ln\left(3-2\sqrt{2}\right)\right)\approx -2.582086$$

ANSWER

$$\int_0^1 \frac{3x^2 - x^2 + 2x - 4}{\sqrt{x^2 - 3x + 2}}\,dx = -\frac{1}{4}\cdot\left(26\sqrt{2} + 15\cdot\ln\left(3 - 2\sqrt{2}\right)\right) \approx -2.582086$$

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