Answer on Question #78062 – Math – Differential Equations

Question

A particle of mass m falls freely under gravity in a liquid that offers a resistive force proportional to its velocity:

$$f_{res} = -\gamma \frac{dx}{dt}$$

Solve it.

Solution

The net force is

$$F_{net} = mg - \gamma \frac{dx}{dt}$$

The differential equation from Newton's Law

$$F_{net} = ma$$

$$a = \frac{d^2x}{dt^2} = g - \frac{\gamma}{m}\frac{dx}{dt}$$

Let $v(t) = \frac{dx}{dt}$ .Then $a = \frac{dv}{dt}$

$$\frac{dv}{dt} = g - \frac{\gamma}{m}v$$

$$\frac{dv}{g - \frac{\gamma}{m}v} = dt$$

$$\int \frac{1}{v - \frac{mg}{\gamma}} dv = -\frac{\gamma}{m}\int dt$$

$$\ln\left|v - \frac{mg}{\gamma}\right| = -\frac{\gamma}{m}t + \ln C$$

$$v - \frac{mg}{\gamma} = Ce^{-\frac{\gamma}{m}t}$$

$$v = \frac{mg}{\gamma} + Ce^{-\frac{\gamma}{m}t}$$

When $t = 0, v(0) = v_0$

$$v_0 = \frac{mg}{\gamma} + C => C = v_0 - \frac{mg}{\gamma}$$

$$v = \frac{mg}{\gamma} + \left(v_0 - \frac{mg}{\gamma}\right)e^{-\frac{\gamma}{m}t}$$

$$\frac{dx}{dt} = v = \frac{mg}{\gamma} + \left(v_0 - \frac{mg}{\gamma}\right)e^{-\frac{\gamma}{m}t}$$

$$dx = \left(\frac{mg}{\gamma} + \left(v_0 - \frac{mg}{\gamma}\right)e^{-\frac{\gamma}{m}t}\right)dt$$

$$x = \frac{mg}{\gamma}t - \frac{m}{\gamma}\left(v_0 - \frac{mg}{\gamma}\right)e^{-\frac{\gamma}{m}t} + C_1$$

When $t = 0, x(0) = x_0$

$$x_0 = -\frac{m}{\gamma}\left(v_0 - \frac{mg}{\gamma}\right)e^{-\frac{\gamma}{m}t} + C_1 => C_1 = x_0 + \frac{m}{\gamma}\left(v_0 - \frac{mg}{\gamma}\right)$$

$$x = \frac{mg}{\gamma}t - \frac{m}{\gamma}\left(v_0 - \frac{mg}{\gamma}\right)e^{-\frac{\gamma}{m}t} + x_0 + \frac{m}{\gamma}\left(v_0 - \frac{mg}{\gamma}\right)$$

$$v = v_0 e^{-\frac{\gamma}{m}t} + \frac{mg}{\gamma}\left(1 - e^{-\frac{\gamma}{m}t}\right)$$

$$x = x_0 + v_0\frac{m}{\gamma}\left(1 - e^{-\frac{\gamma}{m}t}\right) - \frac{m^2 g}{\gamma^2}\left(1 - e^{-\frac{\gamma}{m}t}\right) + \frac{mg}{\gamma}t$$