Question #77772, Math / Differential Geometry | Topology

Compute the normal curvature of the circle α(t)=(cost, sint, 1) on the elliptic paraboloid σ(u, v) = (u, v, u^2 + v^2).

Solution. First we note that $\cos^2 t + \sin^2 t = 1$, so the curve $\boldsymbol{\alpha}(t)$ is contained in the surface $\boldsymbol{\sigma}(u, v) = (u, v, u^2 + v^2)$.

To compute the normal and geodesic curvature of the circle, we need to compute $\boldsymbol{\alpha}'(t), \mathbf{n}(t)$ and $\mathbf{n}(t) \times \boldsymbol{\alpha}'(t)$. In fact, $\boldsymbol{\alpha}'(t) = (-\sin t, \cos t, 0)$, $\|\boldsymbol{\alpha}'(t)\| = 1$, so $\boldsymbol{\alpha}$ is the arc-length parametrization. To find $\mathbf{n}(t)$, we note that $\mathbf{n}(t)$ is the restriction of $\mathbf{n}$ to the curve $\boldsymbol{\alpha}$. So we first calculate $\mathbf{n}$. Since $\boldsymbol{\sigma}_u = (1, 0, 2u), \boldsymbol{\sigma}_v = (0, 1, 2v)$, $\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v = (-2u, -2v, 1)$, $\|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\| = \sqrt{1 + 4u^2 + 4v^2}$. Hence

$$\mathbf{n} = \left. -\frac{2u}{\sqrt{1+4u^2+4v^2}}, -\frac{2v}{\sqrt{1+4u^2+4v^2}}, \frac{1}{\sqrt{1+4u^2+4v^2}} \right).$$

To find $\mathbf{n}(t)$ (the restriction of $\mathbf{n}$ to the curve $\boldsymbol{\alpha}$), we we need to write $\boldsymbol{\alpha}(t) = \boldsymbol{\sigma}(u(t), v(t))$ (since the curve $\boldsymbol{\alpha}$ is contained in the surface, we can always do so). In fact, $\boldsymbol{\alpha}(t) = \boldsymbol{\sigma}(u(t), v(t))$ means that

$$(\cos t, \sin t, 1) = (u(t), v(t), u^2(t) + v^2(t)).$$

This implies that $u(t) = \cos t, v(t) = \sin t$. Since

$$\mathbf{n} = (-\frac{2u}{\sqrt{1+4u^2+4v^2}}, -\frac{2v}{\sqrt{1+4u^2+4v^2}}, \frac{1}{\sqrt{1+4u^2+4v^2}}).$$

The restriction of $\mathbf{n}$ to the curve $\boldsymbol{\alpha}$ is (taking $u(t) = \cos t, v(t) = \sin t$)

$$\mathbf{n}(t) = \mathbf{n}(\boldsymbol{\alpha}(t)) = \left. -\frac{2}{\sqrt{5}} \cos t, -\frac{2}{\sqrt{5}} \sin t, \frac{1}{\sqrt{5}} \right).$$

Finally, $\mathbf{n}(t) \times \boldsymbol{\alpha}'(t) = \left(-\frac{1}{\sqrt{5}} \cos t, -\frac{1}{\sqrt{5}} \sin t, -\frac{1}{\sqrt{5}}\right)$.

To find the normal curvature $\kappa_n(t)$, we note that $\kappa_n(t) = \boldsymbol{\alpha}''(t) \cdot \mathbf{n}(t)$. Since $\boldsymbol{\alpha}''(t) = (-\cos t, -\sin t, 0)$, we have

$$\kappa_n(t) = \boldsymbol{\alpha}''(t) \cdot \mathbf{n}(t) = \frac{2}{\sqrt{5}}.$$