Answer on Question #77661 – Math – Differential Geometry | Topology

Question

Compute the first fundamental form and second fundamental form of the elliptical paraboloid σ(u,v)=(u, v, 〖u〗^2+v^2)

Solution

$$\sigma_u = (1,0,2u),\ \sigma_v = (0,1,2v),$$

$$E = \sigma_u^2 = (1,0,2u)^2 = 1 + 4u^2$$

$$F = \sigma_u \sigma_v = (1,0,2u)(0,1,2v) = 4vu$$

$$G = \sigma_v^2 = (0,1,2v)^2 = 1 + 4v^2$$

First fundamental form: $I = (1 + 4u^2)du^2 + 8uvdudv + (1 + 4v^2)dv^2$.

$$\sigma_u \times \sigma_v = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = \begin{vmatrix} 0 & 2u \\ 1 & 2v \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} 1 & 2u \\ 0 & 2v \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \boldsymbol{k} = -2u\boldsymbol{i} - 2v\boldsymbol{j} + \boldsymbol{k} = (-2u, -2v, 1)$$

$$n = \frac{\sigma_u \times \sigma_v}{\sqrt{EG - F^2}} = \left(-\frac{2u}{\sqrt{1 + 4u^2 + 4v^2}}, -\frac{2v}{\sqrt{1 + 4u^2 + 4v^2}}, \frac{1}{\sqrt{1 + 4u^2 + 4v^2}}\right)$$

$$\sigma_{uu} = (0,0,2),\ \sigma_{uv} = (0,0,0),\ \sigma_{vv} = (0,0,2),$$

$$L = (\sigma_{uu}, n) = \frac{2}{\sqrt{1 + 4u^2 + 4v^2}}$$

$$M = (\sigma_{uv}, n) = 0$$

$$N = (\sigma_{vv}, n) = \frac{2}{\sqrt{1 + 4u^2 + 4v^2}}$$

Second fundamental form: II=$\frac{2}{\sqrt{1+4u^2+4v^2}}\, du^2 + \frac{2}{\sqrt{1+4u^2+4v^2}}\, dv^2$.

Answer: $I = (1 + 4u^2)du^2 + 8uvdudv + (1 + 4v^2)dv^2$, II=$\frac{2}{\sqrt{1+4u^2+4v^2}}\, du^2 + \frac{2}{\sqrt{1+4u^2+4v^2}}\, dv^2$.