Answer on Question #76812 – Math – Abstract Algebra

Question

Define a relation R on Z by $R = \{(n, n + 3k) | k \in Z\}$. Check whether R is an equivalence relation or not. If it is, find all the distinct equivalence classes. If R is not an equivalence relation, define an equivalence relation on Z.

Solution

The relation R is an equivalence relation, if it is reflexive, symmetric and transitive relation. Let's check it.

For any integer $n \in Z$, we have $(n, n) = (n, n + 3 \cdot 0) \in R$, where $k = 0$. This means that R is reflexive.

If $n \in Z$ and $(n, n + 3k) \in R$ for some $k \in Z$, then $(n + 3k, n) = \big(n + 3k, n + 3k + 3(-k)\big) \in R$. This means that R is symmetric.

If $(n, m) \in R$ and $(m, l) \in R$, then there exists integers k_1 and k_1 in Z, such that $m = n + 3k_1$ and $l = m + 3k_2$. Therefore, $l = m + 3(k_1 + k_2)$, i.e. $(m, l) \in R$. The last statement means that R is transitive.

Hence, the relation R on Z is an equivalence relation.

Now find all distinct equivalence classes of the relation R.

$[0] = \{n \in Z: (0, n) \in R, i.e. n = 3k,\ k \in Z\} = \{3k: k \in Z\} = 3Z;$

$[1] = \{n \in Z: (1, n) \in R, i.e. n = 1 + 3k,\ k \in Z\} = \{3k + 1: k \in Z\} = 3Z+1;$

$[2] = \{n \in Z: (2, n) \in R, i.e. n = 2 + 3k,\ k \in Z\} = \{3k + 2: k \in Z\} = 3Z + 2;$

Answer: The relation R is an equivalence relation on Z and there is only three distinct equivalence classes of R: $[0] = \{3k: k \in Z\}$; $[1] = \{3k + 1: k \in Z\}$; $[2] = \{3k + 2: k \in Z\}$.