Answer on Question #76736 – Math – Quantitative Methods

Question

Apply Runge Kutta Method to solve an I.V.P. complicated problem.

Solution

The motion of a material point of mass m under the action of external forces is described by Newton's second law. Let the point move along the x axis, then the function x(t) is the position of the point at time t, it satisfies the ordinary differential equation of the second order mx'' = F (t, x, x'). For example, balance of forces for damped harmonic oscillators:

$$x'' + 2x' + 10x = 0, \qquad x(0) = 2, \qquad x'(0) = 5$$

The exact solution:

$$x(t) = e^{-t}\left(\frac{7}{3}\sin(3t) + 2\cos(3t)\right)$$

For a numerical solution, we transform the second-order differential equation into a system of first-order differential equations by means of the change of variables:

$$y0(t) = x(t), \qquad y1(t) = x'(t)$$

$$\begin{cases} y0'(t) = x'(t) = y1(t) \\ y1'(t) = x''(t) = -2 \cdot y1(t) - 10 \cdot y0(t) \end{cases}, \qquad x(0) = y0(0) = 2, \qquad x'(0) = y1(0) = 5$$

Approximations calculated using the Runge-Kutta method of order 4. For this method, each step requires the use of the following four values (h-step size):

$$k_1 = h \cdot f(t_n, y_n)$$

$$k_2 = h \cdot f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1\right)$$

$$k_3 = h \cdot f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2\right)$$

$$k_4 = h \cdot f(t_n + h, y_n + k_3)$$

Then we calculate the approximation to the solution:

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

For the system of equations:

$$\begin{cases} y_{n+1}^0 = y_n^0 + \frac{1}{6}(k_1^0 + 2k_2^0 + 2k_3^0 + k_4^0) \\ y_{n+1}^1 = y_n^1 + \frac{1}{6}(k_1^1 + 2k_2^1 + 2k_3^1 + k_4^1) \end{cases},$$

$$k_1^0 = h \cdot y_n^1$$

$$k_2^0 = h \cdot \left(y_n^1 + \frac{1}{2}k_1^1\right)$$

$$k_3^0 = h \cdot \left(y_n^1 + \frac{1}{2}k_2^1\right)$$

$$k_4^0 = h \cdot (y_n^1 + k_3^1)$$

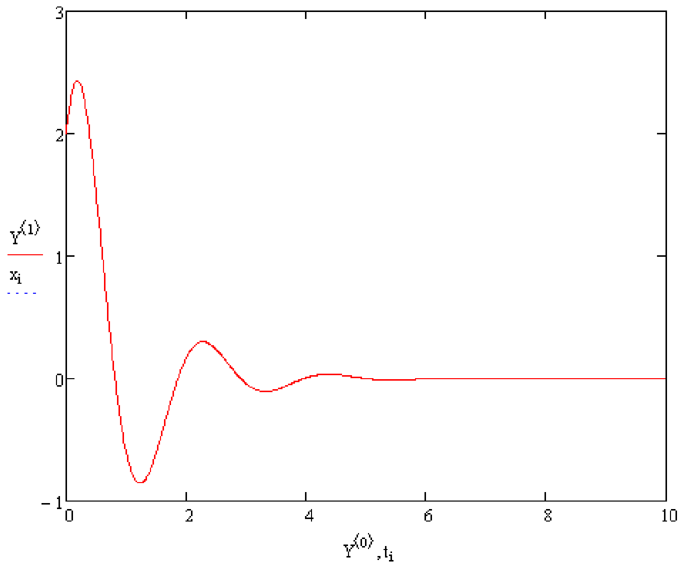
$$k_1^1 = h \cdot (-2 \cdot y_n^1 - 10 \cdot y_n^0)$$

$$k_2^1 = h \cdot \left(-2 \cdot \left(y_n^1 + \frac{1}{2}k_1^1\right) - 10 \cdot \left(y_n^0 + \frac{1}{2}k_1^0\right)\right)$$

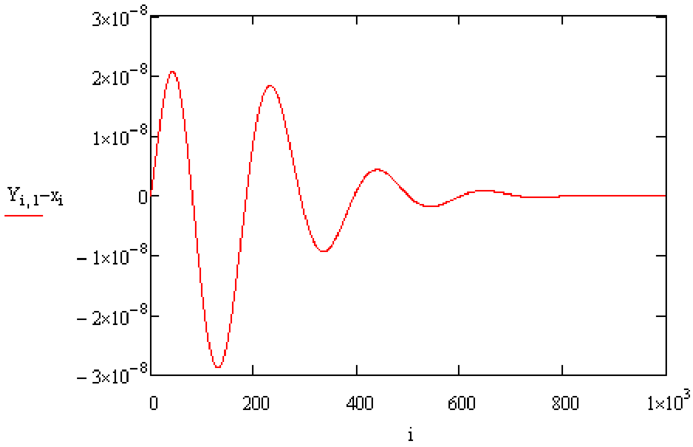
$$k_3^1 = h \cdot \left(-2 \cdot \left(y_n^1 + \frac{1}{2}k_2^1\right) - 10 \cdot \left(y_n^0 + \frac{1}{2}k_2^0\right)\right)$$

$$k_4^1 = h \cdot (-2 \cdot (y_n^1 + k_3^1) - 10 \cdot (y_n^0 + k_3^0))$$

Exact (x_i) and approximate (Y) solutions (h=0.01):



The difference of solutions:


3×10⁻⁸
2×10⁻⁸
1×10⁻⁸
0
− 1×10⁻⁸
− 2×10⁻⁸
− 3×10⁻⁸
Y_{i,1}−x_i
0
200
400
600
800
1×10³
i