Answer on Question #76732 – Math – Quantitative Methods

Question

How to control size of h in Runge Kutta Fehlberg Method?

Solution

To approximate the solution to the 1st order IVP:

$$y' = f(x, y), \qquad y(x_0) = y_0$$

we seek:

$$y_{n+1} = y_n + h \sum_{i=1}^{s} b_i k_i + O(h^{s+1})$$

The adaptive method is designed to produce an estimate of the local truncation error of a single Runge–Kutta step. Let y_{n+1}^{p} and y_{n+1}^{p+1} be the approximations of y_{n+1} computed using the methods of order p and p+1 respectively. The local truncation error in these two methods is given by

$$\varepsilon_{n+1}^{p} = \frac{y_{n+1} - y_{n+1}^{p}}{h}, \qquad \varepsilon_{n+1}^{p+1} = \frac{y_{n+1} - y_{n+1}^{p+1}}{h}$$

The error between two solutions is

$$\varepsilon_{n+1} = \frac{\left|y_{n+1}^{p+1} - y_{n+1}^{p}\right|}{h}$$

If the two answers are in close agreement ($\varepsilon_{n+1} \le \varepsilon$), the approximation is accepted. If the two answers do not agree to a specified accuracy (ε), the step size is reduced. If the answers agree to more significant digits than required, the step size is increased. Our goal is to determine how to modify h. Because ε_{n+1} is the error of a method that is p-th order accurate, then if we replace h by $\delta \cdot h$, the error is multiplied by δ^{p}. To calculate the new step, we must solve the inequality:

$$\left|\delta^{p} \frac{y_{n+1}^{p+1} - y_{n+1}^{p}}{h}\right| < \varepsilon$$

Solving for δ:

$$\delta < \left(\frac{\varepsilon \cdot h}{\left|y_{n+1}^{p+1} - y_{n+1}^{p}\right|}\right)^{1/p} = \left(\frac{\varepsilon}{\varepsilon_{n+1}}\right)^{1/p}$$

The Runge-Kutta-Fehlberg method is a one-step method with the approximations calculated using the Runge-Kutta method of order 4 and 5. For this method each step requires the use of the following six values:

$$k_1 = h \cdot f(x_k, y_k)$$

$$k_2 = h \cdot f\left(x_k + \frac{1}{4}h, y_k + \frac{1}{4}k_1\right)$$

$$k_3 = h \cdot f\left(x_k + \frac{3}{8}h, y_k + \frac{3}{32}k_1 + \frac{9}{32}k_2\right)$$

$$k_4 = h \cdot f\left(x_k + \frac{12}{13}h, y_k + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3\right)$$

$$k_5 = h \cdot f\left(x_k + h, y_k + \frac{439}{216}k_1 - 8k_2 + \frac{3680}{513}k_3 - \frac{845}{4104}k_4\right)$$

$$k_6 = h \cdot f\left(x_k + \frac{1}{2}h, y_k - \frac{8}{27}k_1 + 2k_2 - \frac{3544}{2565}k_3 + \frac{1859}{4104}k_4 - \frac{11}{40}k_5\right)$$

Then we calculate the approximation to the solution with the help of the method of the fourth order:

$$y_{k+1}^4 = y_k + \frac{25}{216}k_1 + \frac{1408}{2565}k_3 + \frac{2197}{4101}k_4 - \frac{1}{5}k_5, \qquad Error = O(h^4)$$

And the approximation to the solution with the help of the method of the 5th order:

$$y_{k+1}^5 = y_k + \frac{16}{135}k_1 + \frac{6656}{12825}k_3 + \frac{28561}{56430}k_4 - \frac{9}{50}k_5 + \frac{2}{55}k_6, \qquad Error = O(h^5)$$

At each step, two different approximations for the solution are made and compared.

$$\varepsilon_{n+1} = \frac{1}{h}\left|y_{k+1}^4 - y_{k+1}^5\right|$$

$$\delta < \left(\frac{\varepsilon}{\varepsilon_{n+1}}\right)^{1/4}$$