Answer on Question #76731 – Math – Quantitative Methods

Question

How to control error in Runge Kutta Fehlberg Method?

Solution

To approximate the solution to the 1st order IVP:

$$y' = f(x, y), \qquad y(x_0) = y_0$$

we seek:

$$y_{n+1} = y_n + h\sum_{i=1}^{s} b_i\, k_i + O(h^{s+1})$$

The adaptive method is designed to produce an estimate of the local truncation error of a single Runge–Kutta step. Let y^p_{n+1} and y^q_{n+1} be the approximations of $y(x_{n+1})$ computed using the methods of order p and q respectively. The local truncation error in these two methods is given by

$$\varepsilon^p_{n+1} = \frac{y_{n+1} - y^p_{n+1}}{h} = O(h^{p+1}), \qquad \varepsilon^q_{n+1} = \frac{y_{n+1} - y^q_{n+1}}{h} = O(h^{q+1})$$

The error between two solutions is

$$\varepsilon_{n+1} = \frac{\left|y^q_{n+1} - y^p_{n+1}\right|}{h}$$

If they differ by no more than ε - the required error, then the approximation is accepted. This method estimates the error of the lower order scheme.

The Runge-Kutta-Fehlberg method is a one-step method with the approximations calculated using the Runge-Kutta method of order 4 and 5. For this method each step requires the use of the following six values:

$$k_1 = h \cdot f(x_k, y_k)$$

$$k_2 = h \cdot f\left(x_k + \frac{1}{4}h, y_k + \frac{1}{4}k_1\right)$$

$$k_3 = h \cdot f\left(x_k + \frac{3}{8}h, y_k + \frac{3}{32}k_1 + \frac{9}{32}k_2\right)$$

$$k_4 = h \cdot f\left(x_k + \frac{12}{13}h, y_k + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3\right)$$

$$k_5 = h \cdot f\left(x_k + h, y_k + \frac{439}{216}k_1 - 8k_2 + \frac{3680}{513}k_3 - \frac{845}{4104}k_4\right)$$

$$k_6 = h \cdot f\left(x_k + \frac{1}{2}h, y_k - \frac{8}{27}k_1 + 2k_2 - \frac{3544}{2565}k_3 + \frac{1859}{4104}k_4 - \frac{11}{40}k_5\right)$$

Then we calculate the approximation to the solution with the help of the method of the fourth order:

$$y_{k+1}^{4} = y_k + \frac{25}{216}k_1 + \frac{1408}{2565}k_3 + \frac{2197}{4101}k_4 - \frac{1}{5}k_5, \qquad Error = O(h^4)$$

And the approximation to the solution with the help of the method of the 5th order:

$$y_{k+1}^{5} = y_k + \frac{16}{135}k_1 + \frac{6656}{12825}k_3 + \frac{28561}{56430}k_4 - \frac{9}{50}k_5 + \frac{2}{55}k_6, \qquad Error = O(h^5)$$

At each step, two different approximations for the solution are made and compared.

$$\varepsilon_{n+1} = \frac{1}{h}\left|y_{k+1}^{4} - y_{k+1}^{5}\right|$$