Question 1. *If (a_n) and (b_n) are Cauchy sequences, show directly that (a_nb_n) is also a Cauchy sequence.*

Solution. First prove that each Cauchy sequence (x_n) is bounded. Fix $\varepsilon > 0$. By the definition of Cauchy sequence there is $N \in \mathbb{N}$ such that $|x_n - x_m| < \varepsilon$ for all $m, n \geq N$. Then for all $n \geq N$ we have

$$|x_n| = |x_n - x_N + x_N| \leq |x_n - x_N| + |x_N| < \varepsilon + |x_N|.$$

Therefore, $|x_n| \leq \max\{|x_1|, \ldots, |x_{N-1}|, |x_N| + \varepsilon\} < \infty$.

Now let (a_n) and (b_n) be Cauchy sequences. There are $A, B > 0$ such that $|a_n| < A$ and $|b_n| < B$ for all $n \in \mathbb{N}$, as shown above. Fix $\varepsilon > 0$ and find $N_1, N_2 \in \mathbb{N}$ such that $|a_n - a_m| < \frac{\varepsilon}{2B}$ for all $m, n > N_1$ and $|b_n - b_m| < \frac{\varepsilon}{2A}$ for all $m, n > N_2$. Set $N = \max\{N_1, N_2\}$. Then for all $m, n > N$ we have

$$\begin{aligned} |a_nb_n - a_mb_m| &= |a_n(b_n - b_m) + b_m(a_n - a_m)| \\ &\leq |a_n| \cdot |b_n - b_m| + |b_m| \cdot |a_n - a_m| \\ &< A \cdot \frac{\varepsilon}{2A} + B \cdot \frac{\varepsilon}{2B} \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Thus, (a_nb_n) is Cauchy sequence. □