ANSWER on Question #76480 – Math – Differential Equations

QUESTION

Obtain a solution of the wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = 16 \cdot \frac{\partial^2 u(x,t)}{\partial x^2}$$

for $0 \le x \le \pi$ and $t > 0$ and the following boundary and initial conditions:

$$\begin{cases} u(0,t) = 0 \\ u(\pi,t) = 0 \end{cases} - \textit{boundary conditions}$$

$$\begin{cases} u(x,0) = x(\pi - x) \\ \dfrac{\partial u(x,0)}{\partial t} = 0 \end{cases} - \textit{initial conditions}$$

SOLUTION

0 STEP: separation of variables.

Let

$$u(x,t) = X(x)T(t) \to \begin{cases} \dfrac{\partial^2 u(x,t)}{\partial t^2} = \dfrac{\partial^2}{\partial t^2}\big(X(x)T(t)\big) = X(x) \cdot \dfrac{d^2\big(T(t)\big)}{dt^2} = X(x) \cdot T''(t) \\ \dfrac{\partial^2 u(x,t)}{\partial x^2} = \dfrac{\partial^2}{\partial x^2}\big(X(x)T(t)\big) = T(t) \cdot \dfrac{d^2\big(X(x)\big)}{dx^2} = X''(x) \cdot T(t) \end{cases}$$

Boundary conditions:

$$\begin{cases} u(0,t) = 0 \\ u(\pi,t) = 0 \end{cases} \to \begin{cases} u(0,t) = X(0)T(t) = 0, \forall t > 0 \\ u(\pi,t) = X(\pi)T(t) = 0, \forall t > 0 \end{cases} \to \boxed{\begin{cases} X(0) = 0 \\ X(\pi) = 0 \end{cases}}$$

Then,

$$\frac{\partial^2 u(x,t)}{\partial t^2} = 16 \cdot \frac{\partial^2 u(x,t)}{\partial x^2} \to X(x) \cdot T''(t) = 16 \cdot X''(x) \cdot T(t) \Big| \times \frac{1}{16X(x)T(t)} \to$$

$$\frac{X(x) \cdot T''(t)}{16X(x)T(t)} = \frac{16 \cdot X''(x) \cdot T(t)}{16X(x)T(t)} \to \boxed{\frac{1}{16} \cdot \frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda}$$

1 STEP: We solve the Sturm-Liouville problem.

(More information: https://en.wikipedia.org/wiki/Sturm%E2%80%93Liouville_theory)

In our case,

$$\begin{cases} \frac{X''(x)}{X(x)} = -\lambda \\ X(0) = 0 \\ X(\pi) = 0 \end{cases}$$

$$\frac{X''(x)}{X(x)} = -\lambda \to X''(x) = -\lambda X(x) \to X''(x) + \lambda X(x) = 0$$

Let us find the solutions of the given equation in the form

$$X(x) = e^{kx} \to X''(x) = k^2 \cdot e^{kx}$$

Then,

$$X''(x) + \lambda X(x) = 0 \to k^2 \cdot e^{kx} + \lambda e^{kx} = 0 \to e^{kx}(k^2 + \lambda) = 0 \to k^2 = -\lambda$$

$$k^2 = -\lambda \to \left[\begin{array}{l} k_1 = \sqrt{-\lambda} = i\sqrt{\lambda} \\ k_2 = -\sqrt{-\lambda} = -i\sqrt{\lambda} \end{array}\right.$$

Then,

$$X(x) = C_1 e^{i\sqrt{\lambda}x} + C_2 e^{-i\sqrt{\lambda}x} \equiv A_1 \cos\left(\sqrt{\lambda}x\right) + A_2 \sin\left(\sqrt{\lambda}x\right)$$

$$\boxed{X(x) = A_1 \cos\left(\sqrt{\lambda}x\right) + A_2 \sin\left(\sqrt{\lambda}x\right)}$$

$$X(0) = 0 = A_1 \cos\left(\sqrt{\lambda} \cdot 0\right) + A_2 \sin\left(\sqrt{\lambda} \cdot 0\right) = A_1 \cos(0) + A_2 \sin(0) = A_1 \cdot 1 + A_2 \cdot 0 \to$$

$$\boxed{A_1 = 0}$$

$$X(\pi) = 0 = A_2 \sin\left(\sqrt{\lambda}\pi\right) \to \sin\left(\sqrt{\lambda}\pi\right) = 0 \to \sqrt{\lambda}\pi = \pi n, n = 1,2,3, \ldots$$

$$\boxed{\lambda_n = n^2, n = 1,2,3, \ldots}$$

Conclusion,

$$\boxed{\begin{cases} X_n(x) = A \cdot \sin(nx) \\ \lambda_n = n^2 \\ n = 1,2,3,\dots \end{cases}}$$

2 STEP: Finding the general solution.

$$\frac{1}{16} \cdot \frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda_n \to \frac{1}{16} \cdot \frac{T''(t)}{T(t)} = -n^2 \to T''(t) = -16n^2 T(t) \to$$

$$T''(t) + 16n^2 T(t) = 0$$

Let us find the solutions of the given equation in the form

$$T(t) = e^{kt} \to T''(t) = k^2 \cdot e^{kt}$$

Then,

$$T''(t) + 16n^2 T(t) = 0 \to k^2 \cdot e^{kt} + 16n^2 e^{kt} = 0 \to e^{kt}(k^2 + 16n^2) = 0 \to k^2 = -16n^2$$

$$k^2 = -16n^2 \to \left[\begin{array}{l} k_1 = \sqrt{-16n^2} = 4in \\ k_2 = -\sqrt{-16n^2} = -4in \end{array}\right.$$

Then,

$$T_n(x) = C_1 e^{4int} + C_2 e^{-4int} \equiv A_1 \cos(4nt) + A_2 \sin(4nt)$$

$$\boxed{T_n(x) = A_n^{(1)} \cos(4nt) + A_n^{(1)} \sin(4nt)}$$

Then,

$$u_n(x,t) = X_n(x) \cdot T_n(t) = (A \cdot \sin(nx)) \cdot \left(A_n^{(1)} \cos(4nt) + A_n^{(1)} \sin(4nt)\right) \to$$

$$\boxed{u_n(x,t) = \left(B_n^{(1)} \cos(4nt) + B_n^{(2)} \sin(4nt)\right) \sin(nx) - particular\ solution}$$

Conclusion,

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) - general\ solution$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \left(B_n^{(1)} \cos(4nt) + B_n^{(2)} \sin(4nt)\right) \sin(nx)}$$

3 STEP: Determine the coefficients B_1 and B_2.

To do this, you must use the initial conditions.

$$\frac{\partial u(x,0)}{\partial t} = 0 \rightarrow$$

$$\frac{\partial}{\partial t}\left(\sum_{n=1}^{\infty} \left(B_n^{(1)} \cos(4nt) + B_n^{(2)} \sin(4nt)\right) \sin(nx)\right)_{t=0} =$$

$$= \left(\sum_{n=1}^{\infty} \left(-4nB_n^{(1)} \sin(4nt) + 4nB_n^{(2)} \cos(4nt)\right) \sin(nx)\right)_{t=0} =$$

$$= \sum_{n=1}^{\infty} \left(-4nB_n^{(1)} \sin(4n \cdot 0) + 4nB_n^{(2)} \cos(4n \cdot 0)\right) \sin(nx) =$$

$$= \sum_{n=1}^{\infty} \left(-4nB_n^{(1)} \cdot 0 + 4nB_n^{(2)} \cdot 1\right) \sin(nx) = \sum_{n=1}^{\infty} 4nB_n^{(2)} \sin(nx) = 0$$

Then,

$$4nB_n^{(2)} = 0 \rightarrow \boxed{B_n^{(2)} = 0}$$

Conclusion,

$$u(x,t) = \sum_{n=1}^{\infty} B_n^{(1)} \cos(4nt) \sin(nx)$$

$$u(x,0) = x(\pi - x) \rightarrow$$

$$u(x,0) = \left(\sum_{n=1}^{\infty} B_n^{(1)} \cos(4nt)\sin(nx)\right)_{t=0} = \sum_{n=1}^{\infty} B_n^{(1)} \cos(4n \cdot 0)\sin(nx) =$$

$$= \sum_{n=1}^{\infty} B_n^{(1)} \cdot 1 \cdot \sin(nx) \rightarrow \sum_{n=1}^{\infty} B_n^{(1)} \sin(nx) = x(\pi - x)$$

As we know

$$\int_0^{\pi} \sin(mx) \cdot \sin(nx)\, dx = \begin{cases} \frac{\pi}{2}, n = m \\ 0, n \neq m \end{cases}$$

In our case,

$$\int_0^{\pi} \times \left| \sum_{n=1}^{\infty} B_n^{(1)} \sin(nx) = x(\pi - x) \right| \times \sin(mx)\, dx$$

$$B_m^{(1)} = \int_0^{\pi} x(\pi - x)\sin(mx)\, dx = \int_0^{\pi} (\pi x - x^2)\sin(mx)\, dx \rightarrow$$

$$B_m^{(1)} = \int_0^{\pi} \pi x \sin(mx)\, dx - \int_0^{\pi} x^2 \sin(mx)\, dx = I_1 - I_2$$

$$I_1 = \int_0^{\pi} \pi x \sin(mx)\, dx = \pi \cdot \int_0^{\pi} \underbrace{x}_{u} \cdot \underbrace{\sin(mx)\, dx}_{dv} = \begin{bmatrix} u = x \to du = dx \\ dv = \sin(mx)\, dx \\ v = \dfrac{-\cos(mx)}{m} \end{bmatrix} =$$

$$= \pi \cdot \left(-\frac{x \cdot \cos(mx)}{m} \Bigg|_0^{\pi} - \int_0^{\pi} \frac{-\cos(mx)\, dx}{m} \right) =$$

$$= \pi \cdot \left(-\frac{\pi \cdot \cos(\pi m)}{m} - \left(-\frac{0 \cdot \cos(m \cdot 0)}{m} \right) + \frac{1}{m} \int_0^{\pi} \cos(mx)\, dx \right) =$$

$$= \pi \cdot \left(-\frac{\pi \cdot (-1)^m}{m} + \frac{1}{m} \cdot \frac{\sin(mx)}{m} \Bigg|_0^{\pi} \right) = \pi \cdot \left(-\frac{\pi \cdot (-1)^m}{m} + \frac{\sin(m \cdot \pi)}{m^2} - \frac{\sin(m \cdot 0)}{m^2} \right) =$$

$$= \pi \cdot \left(-\frac{\pi \cdot (-1)^m}{m} + 0 - 0 \right) = -\frac{\pi^2 \cdot (-1)^m}{m}$$

Conclusion,

$$\boxed{I_1 = -\frac{\pi^2 \cdot (-1)^m}{m}}$$

$$I_2 = \int_0^\pi \underbrace{x^2}_{u} \cdot \underbrace{\sin(mx)\, dx}_{dv} = \begin{bmatrix} u = x^2 \to du = 2xdx \\ dv = \sin(mx)\, dx \\ v = \dfrac{-\cos(mx)}{m} \end{bmatrix} =$$

$$= -\frac{x^2 \cdot \cos(mx)}{m}\Bigg|_0^\pi - \int_0^\pi \frac{-2x \cos(mx)\, dx}{m} =$$

$$= -\frac{\pi^2 \cdot \cos(m\pi)}{m} - \left(-\frac{0^2 \cdot \cos(m \cdot 0)}{m}\right) + \frac{2}{m} \cdot \int_0^\pi x \cos(mx)\, dx =$$

$$= -\frac{\pi^2 \cdot (-1)^m}{m} + \frac{2}{m} \cdot \int_0^\pi \underbrace{x}_{u} \cdot \underbrace{\cos(mx)\, dx}_{dv} = \begin{bmatrix} u = x \to du = dx \\ dv = \cos(mx)\, dx \\ v = \dfrac{\sin(mx)}{m} \end{bmatrix} =$$

$$= -\frac{\pi^2 \cdot (-1)^m}{m} + \frac{2}{m} \cdot \left(\frac{x \cdot \sin(mx)}{m}\Bigg|_0^\pi - \int_0^\pi \frac{\sin(mx)}{m} dx\right) =$$

$$= -\frac{\pi^2 \cdot (-1)^m}{m} + \frac{2}{m} \cdot \left(\frac{\pi \cdot \sin(m\pi)}{m} - \frac{0 \cdot \sin(m \cdot 0)}{m} - \frac{1}{m}\int_0^\pi \sin(mx)\, dx\right) =$$

$$= -\frac{\pi^2 \cdot (-1)^m}{m} + \frac{2}{m} \cdot \left(0 - 0 - \frac{1}{m} \cdot \left(-\frac{\cos(mx)}{m}\Bigg|_0^\pi\right)\right) =$$

$$= -\frac{\pi^2 \cdot (-1)^m}{m} + \frac{2}{m^3} \cdot (\cos(m\pi) - \cos(m \cdot 0)) =$$

$$= -\frac{\pi^2 \cdot (-1)^m}{m} + \frac{2}{m^3} \cdot ((-1)^m - 1)$$

$$\boxed{I_2 = -\frac{\pi^2 \cdot (-1)^m}{m} + \frac{2}{m^3} \cdot ((-1)^m - 1)}$$

Then,

$$B_m^{(1)} = I_1 - I_2 = -\frac{\pi^2 \cdot (-1)^m}{m} - \left(-\frac{\pi^2 \cdot (-1)^m}{m} + \frac{2}{m^3} \cdot ((-1)^m - 1)\right) \rightarrow$$

$$B_m^{(1)} = -\frac{\pi^2 \cdot (-1)^m}{m} + \frac{\pi^2 \cdot (-1)^m}{m} - \frac{2}{m^3} \cdot ((-1)^m - 1) \rightarrow$$

$$\boxed{B_m^{(1)} = \frac{-2 \cdot ((-1)^m - 1)}{m^3}}$$

As we know

$$\begin{cases} (-1)^m - 1 = 0, m = 2k, k = 1,2,3,4, \dots \\ (-1)^m - 1 = -2, m = 2k - 1, k = 1,2,3,4, \dots \end{cases} \rightarrow$$

$$\begin{cases} B_m^{(1)} = 0, m = 2k, k = 1,2,3,4, \dots \\ B_m^{(1)} = \dfrac{4}{m^3}, m = 2k - 1, k = 1,2,3,4, \dots \end{cases}$$

Conclusion,

$$u(x,t) = \sum_{n=1}^{\infty} B_n^{(1)} \cos(4nt) \sin(nx) = \sum_{n=1}^{\infty} \left(\frac{-2 \cdot ((-1)^n - 1)}{n^3}\right) \cos(4nt) \sin(nx)$$

$$\boxed{u(x,t) = \sum_{k=1}^{\infty} \left(\frac{4}{(2k-1)^3}\right) \cos(4(2k-1)t) \sin\big((2k-1)x\big)}$$

ANSWER:

$$u(x,t) = \sum_{k=1}^{\infty} \left(\frac{4}{(2k-1)^3}\right) \cos(4(2k-1)t) \sin\big((2k-1)x\big)$$