Answer on Question #76479 – Math – Differential Equations

Question

Reduce the following PDE to a set of three ODEs by the method of separation of variables

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\left(\frac{\partial V}{\partial r}\right)\right)+\left(\frac{1}{r^2}\right)\left(\frac{\partial^2 V}{\partial \theta^2}\right)+\left(\frac{\partial^2 V}{\partial z^2}\right)+(k^2)V=0$$

Solution

Assume

$$V(r,\theta,z)=R(r)\Theta(\theta)Z(z)$$

$$\Theta Z\left(\frac{1}{r}\right)\frac{d}{dr}\left(r\left(\frac{dR}{dr}\right)\right)+RZ\left(\frac{1}{r^2}\right)\left(\frac{d^2\Theta}{d\theta^2}\right)+R\Theta\left(\frac{d^2 Z}{dz^2}\right)+(k^2)R\Theta Z=0$$

Divide through by V:

$$\frac{1}{R}\left(\frac{1}{r}\right)\frac{d}{dr}\left(r\left(\frac{dR}{dr}\right)\right)+\frac{1}{\Theta}\left(\frac{1}{r^2}\right)\left(\frac{d^2\Theta}{d\theta^2}\right)+\frac{1}{Z}\left(\frac{d^2 Z}{dz^2}\right)+k^2=0$$

$$\frac{1}{R}\left(\frac{1}{r}\right)\frac{dR}{dr}+\frac{1}{R}\frac{d^2R}{dr^2}+\frac{1}{\Theta}\left(\frac{1}{r^2}\right)\left(\frac{d^2\Theta}{d\theta^2}\right)+k^2=-\frac{1}{Z}\left(\frac{\partial^2 Z}{\partial z^2}\right)$$

In the above equation the left-hand side depends on r and θ, while the right-hand side depends on z. The only way these two members are going to be equal for all values of r,θ and z is when both of them are equal to a constant. Let us define such a constant as $-l^2$.

With this choice for the constant, we obtain:

$$\frac{d^2Z}{dz^2}-l^2Z=0$$

The general solution of this equation is:

$$Z(z)=a_1e^{lz}+a_2e^{-lz}$$

Such a solution, when considering the specific boundary conditions, will allow $Z(z)$ to go to zero for z going to $\pm\infty$, which makes physical sense. If we had given the constant a value of l^2, we would have had periodic trigonometric functions, which do not tend to zero for z going to infinity.

Once sorted the z-dependency, we need to take care of r and θ.

$$\frac{1}{R}\left(\frac{1}{r}\right)\frac{dR}{dr}+\frac{1}{R}\frac{d^2R}{dr^2}+\frac{1}{\Theta}\left(\frac{1}{r^2}\right)\left(\frac{d^2\Theta}{d\theta^2}\right)=-(k^2+l^2)$$

$$\frac{r}{R}\frac{dR}{dr}+\frac{r^2}{R}\frac{d^2R}{dr^2}+(k^2+l^2)r^2=-\frac{1}{\Theta}\left(\frac{d^2\Theta}{d\theta^2}\right)$$

Again we are in a situation where the only way a solution can be found for the above equation is when both members are equal to a constant. This time we select a positive constant, which we call m^2. The equation for Θ becomes, then:

$$\frac{d^2\Theta}{d\theta^2} + m^2\Theta = 0$$

Its general solution can be written as:

$$\Theta(\theta) = b_1 \sin(m\theta) + b_2 \cos(m\theta)$$

This solution is well suited to describe the variation for an angular coordinate like θ. Had we chosen to set both members of equation equal to a negative number, we would have ended up with exponential functions with a different value assigned to $\Theta(\theta)$ for each 360 degrees turn, a clear non-physical solution.

Last to be examined is the r-dependency. We have:

$$\frac{r}{R}\frac{dR}{dr} + \frac{r^2}{R}\frac{d^2R}{dr^2} + (k^2 + l^2)r^2 = m^2$$

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + \left((k^2 + l^2)r^2 - m^2\right)R = 0 \qquad (*)$$

The equation $(*)$ is a well-known equation of mathematical physics called parametric Bessel's equation. With a simple linear transformation of variable, $x = (\sqrt{k^2 + l^2})r$, equation $(*)$ is readily changed into a Bessel's equation:

$$\frac{dR}{dr} = \frac{dR}{dx}\frac{dx}{dr} = \left(\sqrt{k^2 + l^2}\right)R'$$

$$\frac{d^2R}{dr^2} = \frac{d}{dx}\left(\left(\sqrt{k^2 + l^2}\right)R'\right)\frac{dx}{dr} = (k^2 + l^2)R''$$

$$\frac{x^2}{k^2 + l^2}(k^2 + l^2)R'' + \frac{x}{\sqrt{k^2 + l^2}}\left(\sqrt{k^2 + l^2}\right)R' + (x^2 - m^2)R = 0$$

$$x^2R'' + xR' + (x^2 - m^2)R = 0$$

where R'' and R' indicate first and second derivatives with respect to x.

In what follows we will assume that m is a real, non-negative number.

Linearly independent solutions are typically denoted $J_m(x)$ (Bessel Functions) and $N_m(x)$ (Neumann Functions).