ANSWER on Question #76436 – Math – Differential Equations

QUESTION

Solve the partial differential equation

$$(D^2 + D - 1)z = 4e^{x+y}\cos(x+y)$$

SOLUTION

Let us use some known facts.

Let the given differential equation be

$$F(D, D') = f(x, y).$$

Factorize $F(D, D')$ into linear factors. Then use the following results:

Rule I. Corresponding to each non-repeated factor $(bD - aD' - c)$, the part of C.F. is taken as

$$e^{\left(\frac{cx}{b}\right)}\varphi(by + ax),\ \ if\ b \neq 0$$

We now have three particular cases of Rule I:

Rule IA. Take $c = 0$ in Rule I. Hence corresponding to each linear factor $(bD - aD')$, the part of C.F. is

$$\varphi(by + ax),\ \ if\ b \neq 0.$$

Rule IB. Take $a = 0$ in Rule I. Hence corresponding to each linear factor $(bD - c)$, the part of C.F. is

$$e^{\left(\frac{cx}{b}\right)}\varphi(by),\ \ if\ b \neq 0.$$

Rule IC. Take $a = c = 0$ and $b = 1$ in Rule I. Hence corresponding to each linear factor $(1 \cdot D)$, the part of C.F. is

$$\varphi(y).$$

Rule II. When

$$f(x,y) = V(x,y)e^{ax+by}$$

Then,

$$P.I. = \frac{1}{F(D,D')}Ve^{ax+by} = e^{ax+by}\frac{1}{F(D+a,D'+b)}V(x,y)$$

In our case,

$$(D^2 + D - 1)z = 4e^{x+y}\cos(x+y) \rightarrow z(x,y) = C.F. + P.I.$$

0 STEP: Factorize $F(D,D')$ into linear factors.

$$D^2 + D - 1 \rightarrow \underbrace{1}_{a} \cdot x^2 + \underbrace{1}_{b} \cdot x \underbrace{-1}_{c} = 0 \rightarrow \sqrt{D} = \sqrt{b^2 - 4ac} = \sqrt{(1)^2 - 4 \cdot 1 \cdot (-1)} = \sqrt{1+4}$$

$$\sqrt{D} = \sqrt{5} \rightarrow \left[\begin{array}{l} m_1 = \dfrac{-b-\sqrt{D}}{2a} = \dfrac{-1-\sqrt{5}}{2\cdot 1} = -\dfrac{1+\sqrt{5}}{2} \\ m_2 = \dfrac{-b+\sqrt{D}}{2a} = \dfrac{-1+\sqrt{5}}{2\cdot 1} = -\dfrac{1-\sqrt{5}}{2} \end{array}\right.$$

Conclusion,

$$m^2 + m - 1 = (m - m_1)(m - m_2) = \left(m + \frac{1+\sqrt{5}}{2}\right)\left(m + \frac{1-\sqrt{5}}{2}\right) \rightarrow$$

$$D^2 + D - 1 = \left(D + \frac{1+\sqrt{5}}{2}\right)\left(D + \frac{1-\sqrt{5}}{2}\right)$$

Then,

$$\boxed{(D^2 + D - 1)z = 4e^{x+y}\cos(x+y) \rightarrow \left(D + \frac{1+\sqrt{5}}{2}\right)\left(D + \frac{1-\sqrt{5}}{2}\right)z = 4e^{x+y}\cos(x+y)}$$

1 STEP: Let find C.F.

$$\begin{cases}\left(D+\frac{1+\sqrt{5}}{2}\right)z \\ (bD-aD'-c)z\end{cases} \rightarrow \begin{cases} b=1 \\ a=0 \\ c=-\frac{1+\sqrt{5}}{2}\end{cases} \rightarrow (C.F.)_1 = e^{\left(\frac{-\frac{1+\sqrt{5}}{2}\cdot x}{1}\right)} \cdot \varphi_1(1\cdot y+(0)\cdot x) \rightarrow$$

$$\boxed{(C.F.)_1 = e^{-\frac{1+\sqrt{5}}{2}\cdot x} \cdot \varphi_1(y), where\ \varphi_1\ is\ arbitrary\ function}$$

$$\begin{cases}\left(D+\frac{1-\sqrt{5}}{2}\right)z \\ (bD-aD'-c)z\end{cases} \rightarrow \begin{cases} b=1 \\ a=0 \\ c=-\frac{1-\sqrt{5}}{2}\end{cases} \rightarrow (C.F.)_2 = e^{\left(\frac{-\frac{1-\sqrt{5}}{2}\cdot x}{1}\right)} \cdot \varphi_2(1\cdot y+(0)\cdot x) \rightarrow$$

$$\boxed{(C.F.)_2 = e^{-\frac{1-\sqrt{5}}{2}x} \cdot \varphi_2(y), where\ \varphi_2\ is\ arbitrary\ function}$$

Then,

$$C.F. = (C.F.)_1 + (C.F.)_2 \rightarrow \boxed{C.F. = e^{-\frac{1+\sqrt{5}}{2}\cdot x} \cdot \varphi_1(y) + e^{-\frac{1-\sqrt{5}}{2}x} \cdot \varphi_2(y)}$$

2 STEP: Let find P.I.

$$P.I. = \frac{1}{D^2+D-1} 4e^{\overset{a}{\widehat{1}}\cdot x+\overset{b}{\widehat{1}}\cdot y}\cos(x+y) = 4e^{x+y}\frac{1}{(D+1)^2+(D+1)-1}\cos(x+y) =$$

$$= 4e^{x+y}\frac{1}{D^2+2D+1+D+1-1}\cos(x+y) = 4e^{x+y}\frac{1}{D^2+3D+1}\cos(x+y) =$$

$$= 4e^{x+y}\frac{1}{(-1)^2+3D+1}\cos(x+y) = 4e^{x+y}\frac{1}{1+3D+1}\cos(x+y) =$$

$$= 4e^{x+y}\frac{1}{3D+2}\cos(x+y) = 4e^{x+y}\frac{(3D-2)}{(3D+2)(3D-2)}\cos(x+y) =$$

$$= 4e^{x+y}\frac{(3D-2)}{9D^2-4}\cos(x+y) = 4e^{x+y}\frac{(3D-2)}{9\cdot(-1)^2-4}\cos(x+y) =$$

$$= 4e^{x+y}\frac{(3D-2)}{9\cdot 1-4}\cos(x+y) = 4e^{x+y}\frac{(3D-2)}{5}\cos(x+y) =$$

$$=\frac{4}{5}e^{x+y}(3D-2)\cos(x+y)=\frac{4}{5}e^{x+y}\left(3\frac{\partial}{\partial x}-2\right)\cos(x+y)=$$

$$=\frac{4}{5}e^{x+y}\left(3\frac{\partial}{\partial x}(\cos(x+y))-2\cdot\cos(x+y)\right)=$$

$$=\frac{4}{5}e^{x+y}(3(-\sin(x+y))-2\cos(x+y))=$$

$$=-\frac{4}{5}e^{x+y}(3\sin(x+y)+2\cos(x+y))$$

Then,

$$\boxed{P.I.=-\frac{4}{5}e^{x+y}(3\sin(x+y)+2\cos(x+y))}$$

Conclusion,

$$z(x,y)=C.F.+P.I.=$$

$$=e^{-\frac{1+\sqrt{5}}{2}\cdot x}\cdot\varphi_1(y)+e^{-\frac{1-\sqrt{5}}{2}x}\cdot\varphi_2(y)-\frac{4}{5}e^{x+y}(3\sin(x+y)+2\cos(x+y))$$

$$z(x,y)=e^{-\frac{1+\sqrt{5}}{2}\cdot x}\cdot\varphi_1(y)+e^{-\frac{1-\sqrt{5}}{2}x}\cdot\varphi_2(y)-\frac{4}{5}e^{x+y}(3\sin(x+y)+2\cos(x+y))$$

$$\boxed{\begin{cases}z(x,y)=e^{-\frac{1+\sqrt{5}}{2}\cdot x}\cdot\varphi_1(y)+e^{-\frac{1-\sqrt{5}}{2}x}\cdot\varphi_2(y)-\frac{4}{5}e^{x+y}(3\sin(x+y)+2\cos(x+y))\\ \textit{where } \varphi_1 \textit{and } \varphi_2 \textit{ are arbitrary functions}\end{cases}}$$

ANSWER:

$$\begin{cases}z(x,y)=e^{-\frac{1+\sqrt{5}}{2}\cdot x}\cdot\varphi_1(y)+e^{-\frac{1-\sqrt{5}}{2}x}\cdot\varphi_2(y)-\frac{4}{5}e^{x+y}(3\sin(x+y)+2\cos(x+y))\\ \textit{where } \varphi_1 \textit{and } \varphi_2 \textit{ are arbitrary functions}\end{cases}$$