Answer on Question #76405 – Math – Calculus

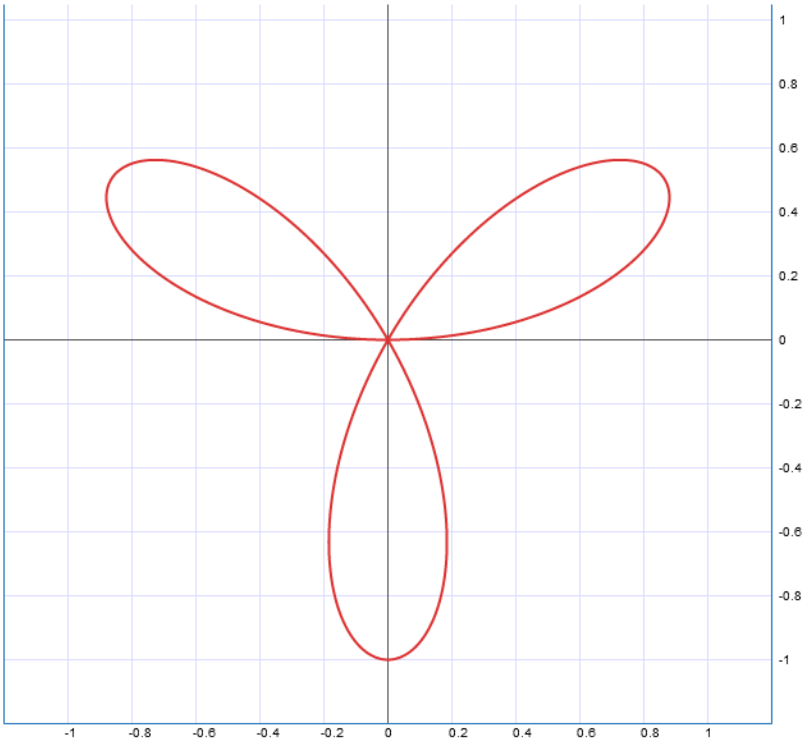
Question

Find the area of a loop of the curve

r = a sin 3theta

Solution

The curve of the polar equation $r = a\sin 3\theta$ at $a = 1$ is



To find the area of one loop we define the angles at which the loop begins and ends. For this we equate $r = a\sin 3\theta$ to zero and solve the equation

$$a\sin 3\theta = 0$$

with respect to θ. We have $3\theta = 0,\ 3\theta = \pi,\ 3\theta = 2\pi, \ldots$ so we get $\theta = 0,\ \theta = \pi/3,\ \theta = 2\pi/3, \ldots$ Thus, the curve in the first quadrant varies from $\theta = 0$ to $\theta = \pi/3$.

The expression for the area of a figure bounded by a curve $r = r(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is given by

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2(\theta) d\theta$$

For one loop of the given equation we have

$$A = \frac{1}{2} \int_{0}^{\pi/3} (a\sin 3\theta)^2 d\theta = \frac{a}{2} \int_{0}^{\pi/3} \sin^2 3\theta d\theta$$

Using the identity $\sin^2\varphi = \frac{1}{2}(1 - \cos 2\varphi)$ we get

$$A = \frac{a}{2}\int_{0}^{\pi/3} \frac{1}{2}(1 - \cos 6\theta)d\theta = \frac{a}{4}\int_{0}^{\pi/3} (1 - \cos 6\theta)d\theta$$

Integration yields

$$A = \frac{a}{4}\left(\theta - \frac{1}{6}\sin 6\theta\right)\Big|_{0}^{\pi/3}$$

$$= \frac{a}{4}\left(\frac{\pi}{3} - \frac{1}{6}\sin 2\pi - \left(0 - \frac{1}{6}\sin 0\right)\right)$$

$$= \frac{a}{4}\left(\frac{\pi}{3}\right)$$

$$= \frac{\pi a}{12}$$

Answer: the area of a loop is

$$A = \frac{\pi a}{12}$$