Answer on Question #76402 – Math – Calculus

Question

Find the area of the loop of the curve, x (x^2+ y^2) = a (x^2- y^2).

Solution

Converting to polar coordinates using $x = rcos\theta$ and $y = rsin\theta$:

$$rcos\theta(r^2cos^2\theta + r^2sin^2\theta) = a(r^2cos^2\theta - r^2sin^2\theta)$$

$$r^3cos\theta(cos^2\theta + sin^2\theta) = ar^2(cos^2\theta - sin^2\theta)$$

Using $(cos^2\theta + sin^2\theta) = 1$:

$$r^3cos\theta = ar^2(cos^2\theta - (1 - cos^2\theta))$$

$$r^2\left(rcos\theta - a(2cos^2\theta - 1)\right) = 0$$

Note that $r^2 = 0$ is just the point (0,0), so we're left with:

$$rcos\theta - a(2cos^2\theta - 1) = 0$$

$$r = a(2\cos\theta - \sec\theta), \text{where } \cos\theta \neq 0$$

The loop will begin and end when $r = 0$, which is when

$$2cos^2\theta = 1$$

$$\cos\theta = \pm\frac{\sqrt{2}}{2}$$

$$\theta = \pm\frac{\pi}{4}$$

The area of a polar curve r from $\theta = \alpha$ to $\theta = \beta$ is given by $\frac{1}{2}\int_\beta^\alpha r^2\, d\theta$, so here the integral for the area is

$$\frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}[a(2\cos\theta - \sec\theta)]^2\, d\theta$$

Since the loop is symmetric across the x axis:

$$= \int_0^{\frac{\pi}{4}} a^2(2\cos\theta - \sec\theta)^2\, d\theta = a^2\int_0^{\frac{\pi}{4}}(4\cos^2\theta - 4\cos\theta\sec\theta + \sec^2\theta)d\theta$$

Using $\cos^2\theta = \frac{1}{2}(1 + cos2\theta)$:

$$= a^2\int_0^{\frac{\pi}{4}}(2(1 + \cos2\theta) - 4 + \sec^2\theta)d\theta = a^2\int_0^{\frac{\pi}{4}}(2\cos2\theta - 2 + \sec^2\theta)d\theta$$

$$= a^2(\sin2\theta - 2\theta + \tan\theta)\Big|_0^{\frac{\pi}{4}}$$

$$= a^2\left(\sin\frac{\pi}{2} - \frac{\pi}{2} + \tan\frac{\pi}{4}\right) - a^2(\sin0 - 0 + \tan0)$$

$$= a^2\left(1 - \frac{\pi}{2} + 1\right)$$

$$= a^2\frac{4 - \pi}{2}$$

Answer: $a^2\frac{4-\pi}{2}$.

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