Answer on Question #76398 – Math – Calculus

Question

Find the volume of the solid generated by the revolution of the curve

$$(a-x)y^2 = a^2x$$

about its asymptote.

Solution

$$y^2 = \frac{a^2x}{a-x}$$

Asymptote is:

$$x = a$$

The volume:

$$V = \pi \int_{-\infty}^{\infty} (f(y) - a)^2 dy$$

$$f(y) = x = \frac{ay^2}{y^2 + a^2}$$

$$V = \pi \int_{-\infty}^{\infty} \left(\frac{ay^2}{y^2 + a^2} - a\right)^2 dy = \pi a^2 \left(\int_{-\infty}^{\infty} \frac{y^4}{(y^2 + a^2)^2} dy - 2 \int_{-\infty}^{\infty} \frac{y^2}{y^2 + a^2} dy + \int_{-\infty}^{\infty} dy\right)$$

$$\int_{-\infty}^{\infty} \frac{y^2}{y^2 + a^2} dy = \int_{-\infty}^{\infty} \left(1 - \frac{a^2}{y^2 + a^2}\right) dy$$

$$\int_{-\infty}^{\infty} \frac{y^4}{(y^2 + a^2)^2} dy = \int_{-\infty}^{\infty} \frac{y^4}{y^4 + 2y^2a^2 + a^4} dy = \int_{-\infty}^{\infty} \left(1 - \frac{2y^2a^2 + a^4}{y^4 + 2y^2a^2 + a^4}\right) dy$$

$$\int_{-\infty}^{\infty} \frac{2y^2a^2 + a^4}{y^4 + 2y^2a^2 + a^4} dy = a^2 \int_{-\infty}^{\infty} \frac{2y^2 + a^2}{(y^2 + a^2)^2} dy$$

$$\frac{2y^2 + a^2}{(y^2 + a^2)^2} = \frac{1}{y^2 + a^2} + \frac{y^2}{(y^2 + a^2)^2} = \frac{2}{y^2 + a^2} - \frac{a^2}{(y^2 + a^2)^2}$$

Then:

$$V = \pi a^2 \left(\int_{-\infty}^{\infty} dy - \int_{-\infty}^{\infty} \frac{2a^2}{y^2 + a^2} dy + \int_{-\infty}^{\infty} \frac{2a^4 dy}{(y^2 + a^2)^2} - 2 \int_{-\infty}^{\infty} dy + \int_{-\infty}^{\infty} \frac{2a^2}{y^2 + a^2} dy + \int_{-\infty}^{\infty} dy \right)$$

$$V = 2\pi a^6 \int_{-\infty}^{\infty} \frac{dy}{(y^2 + a^2)^2}$$

$$\int_{-\infty}^{\infty} \frac{dy}{(y^2 + a^2)^2} \Rightarrow y = a \tan\theta\,;\; dy = \frac{a}{cos^2\theta} d\theta$$

$$\int_{-\infty}^{\infty} \frac{dy}{(y^2 + a^2)^2} = \frac{1}{a^3} \int_{-\pi/2}^{\pi/2} cos^2\theta d\theta = \frac{1}{2a^3} \int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta) d\theta = \frac{1}{2a^3} \left(\theta + \frac{\sin 2\theta}{2} \right) \Big|_{-\pi/2}^{\pi/2} = \frac{\pi}{2a^3}$$

$V = 2\pi a^6 \int_{-\infty}^{\infty} \frac{dy}{(y^2+a^2)^2} = 2\pi a^6 \cdot \frac{\pi}{2a^3} = \pi^2 a^3.$

Answer: $V = \pi^2 a^3$.