Answer on Question #76366 – Math – Discrete Mathematics

Question

Prove that for all positive integers n $3^n + 7^n - 2$ is divisible by 8.

Proof

For $n = 1$ the statement is true, since $3^1 + 7^1 - 2 = 8$ is divisible by 8.

Let $n > 1$. Then we can assume that $n = 2k$ (an even integer) or $n = 2k + 1$ (an odd integer) for some positive integer k.

1) Let first $n = 2k$ for some positive integer k. Consider

$$3^n + 7^n - 2 = 3^{2k} + 7^{2k} - 2 = 9^k + 49^k - 2.$$

By Newton binomial we have that

$$9^k = (8 + 1)^k = 8A + 1$$

and

$$49^k = (6 * 8 + 1)^k = 8B + 1$$

for some positive integers A and B.

Therefore, we have that

$$3^n + 7^n - 2 = 8A + 1 + 8B + 1 - 2 = 8(A + B) \text{ is divisible by 8.}$$

2) Let now $n = 2k + 1$ for some positive integer k (in general, the integer k in this case does not coincide with the previous integer k in the case 1)). Consider

$$3^n + 7^n - 2 = 3^{2k+1} + 7^{2k+1} - 2 = 3 * 9^k + 7 * 49^k - 2.$$

Similarly to case 1) from Newton binomial there exists positive integers C and D, such that

$$9^k = (8 + 1)^k = 8C + 1$$

and

$$49^k = (6 * 8 + 1)^k = 8D + 1.$$

Therefore, we have that

$$3^n + 7^n - 2 = 3(8C + 1) + 7(8D + 1) - 2 = 8(3C + 7D + 1) \text{ is divisible by 8.}$$

All even and odd numbers were analyzed in cases 1), 2), they form the set of integers.

The proof is completed.