ANSWER on Question #76221 – Math – Discrete Mathematics

QUESTION

Prove the following formulas for all positive integers n.

$$\boldsymbol{a})\ 1+2+3+4+5+\ldots+n=\frac{n(n\ +\ 1)}{2}$$

$$\boldsymbol{b})\ 1+4+9+16+25+\ldots+n^2=\frac{n(n\ +\ 1)(2n\ +\ 1)}{6}$$

$\boldsymbol{c})\ 2^{2n}-1$ is a multiple of 3

SOLUTION

All formulas will be proved by using the method of mathematical induction.

(More information: https://en.wikipedia.org/wiki/Mathematical_induction)

a)

$$1+2+3+4+5+\ldots+n=\frac{n(n\ +\ 1)}{2}$$

1 STEP: Basis of induction

We verify the validity of the formula for $n=1$

$$for\ n=1:1+2+3+4+5+\cdots+n=1$$

$$for\ n=1:\frac{n(n+1)}{2}=\frac{1\cdot(1+1)}{2}=\frac{1\cdot 2}{2}=1$$

Conclusion,

$$\begin{cases} for\ n=1:1+2+3+4+5+\cdots+n=1 \\ for\ n=1:\dfrac{n(n+1)}{2}=1 \end{cases}\rightarrow$$

$$\boxed{1+2+3+4+5+\ldots+n=\frac{n(n\ +\ 1)}{2},\ \ for\ n=1}$$

2 STEP: Inductive hypothesis

The formula is true for $1 \leq k \leq n$

$$1 + 2 + 3 + 4 + 5 + \ldots + k = \frac{k(k\ +\ 1)}{2}$$

3 STEP: The inductive step

It is necessary to prove that for $k = n + 1$

$$1 + 2 + 3 + 4 + 5 + \cdots + (n + 1) = \frac{(n + 1)(n + 2)}{2}$$

In our case,

$$1 + 2 + 3 + 4 + 5 + \cdots + (n + 1) = \underbrace{(1 + 2 + 3 + 4 + 5 + \cdots + n)}_{\frac{n(n+1)}{2}} + (n + 1) =$$

$$= \frac{n(n + 1)}{2} + \frac{(n + 1)}{1}^{\backslash 2} = \frac{n(n + 1)}{2} + \frac{2(n + 1)}{2} = \frac{n(n + 1) + 2(n + 1)}{2} = \frac{(n + 1)(n + 2)}{2}$$

Conclusion,

$$1 + 2 + 3 + 4 + 5 + \cdots + (n + 1) = \frac{(n + 1)(n + 2)}{2}$$

Q.E.D.

b)

$$1 + 4 + 9 + 16 + 25 + \ldots + n^2 = \frac{n(n\ +\ 1)(2n\ +\ 1)}{6}$$

1 STEP: Basis of induction

We verify the validity of the formula for $n = 1$

$$for\ n = 1 : 1 + 4 + 9 + 16 + 25 + \cdots + n^2 = 1^2 = 1$$

$$for\ n = 1 : \frac{n(n\ +\ 1)(2n\ +\ 1)}{6} = \frac{1 \cdot (1 + 1) \cdot (2 \cdot 1 + 1)}{6} = \frac{1 \cdot 2 \cdot 3}{6} = 1$$

Conclusion,

$$\begin{cases} for\ n = 1 : 1 + 4 + 9 + 16 + 25 + \cdots + n^2 = 1 \\ for\ n = 1 :: \frac{n(n\ +\ 1)(2n\ +\ 1)}{6} = 1 \end{cases} \rightarrow$$

$$\boxed{1 + 4 + 9 + 16 + 25 + \ldots + n^2 = \frac{n(n\ +\ 1)(2n\ +\ 1)}{6},\ \ for\ n = 1}$$

2 STEP: Inductive hypothesis

The formula is true for $1 \leq k \leq n$

$$1 + 4 + 9 + 16 + 25 +\ \ldots + k^2 = \frac{k(k\ +\ 1)(2k\ +\ 1)}{6}$$

3 STEP: The inductive step

It is necessary to prove that for $k = n + 1$

$$1 + 4 + 9 + 16 + 25 + \cdots + (n + 1)^2 = \frac{(n + 1)(n + 2)(2(n + 1) + 1)}{6}$$

In our case,

$$1 + 4 + 9 + 16 + 25 + \cdots + (n + 1)^2 = \underbrace{(1 + 4 + 9 + 16 + 25 + \cdots + n^2)}_{\frac{n(n+1)(2n+1)}{6}} + (n + 1)^2 =$$

$$= \frac{n(n + 1)(2n + 1)}{6} + \frac{(n + 1)^{2 \backslash 6}}{1} = \frac{n(n + 1)(2n + 1)}{6} + \frac{6(n + 1)^2}{6} =$$

$$= \frac{n(n + 1)(2n + 1) + 6(n + 1)^2}{6} = \frac{(n + 1)\big(n(2n + 1) + 6(n + 1)\big)}{6} =$$

$$= \frac{(n + 1)(2n^2 + n + 6n + 6)}{6} = \frac{(n + 1)(2n^2 + 7n + 6)}{6} = \frac{(n + 1)(2n^2 + 3n + 4n + 6)}{6} =$$

$$= \frac{(n + 1)\big(n(2n + 3) + 2(2n + 3)\big)}{6} = \frac{(n + 1)(2n + 3)(n + 2)}{6} =$$

$$= \frac{(n+1)(n+2)(2(n+1)+1)}{6}$$

Conclusion,

$$1+4+9+16+25+\cdots+(n+1)^2 = \frac{(n+1)(n+2)(2(n+1)+1)}{6}$$

Q.E.D.

c)

$$2^{2n} - 1 \; is\; a\; multiple\; of\; 3$$

1 STEP: Basis of induction

We verify the validity of the formula for $n = 1$

$$for\; n = 1 : 2^{2\cdot 1} - 1 = 2^2 - 1 = 4 - 1 = 3 \; is\; a\; multiple\; of\; 3$$

Conclusion,

$$\boxed{2^{2n} - 1 \; is\; a\; multiple\; of\; 3, \;\; for\; n = 1}$$

2 STEP: Inductive hypothesis

The formula is true for $1 \le k \le n$

$$2^{2k} - 1 \; is\; a\; multiple\; of\; 3$$

3 STEP: The inductive step

It is necessary to prove that for $k = n + 1$

$$2^{2(n+1)} - 1 \; is\; a\; multiple\; of\; 3$$

In our case,

$$2^{2(n+1)} - 1 = 2^{2n+2} - 1 = 2^{2n} \cdot 2^2 - 1 = 2^{2n} \cdot 4 - 1 = 2^{2n} \cdot (3+1) - 1 =$$

$$= 3 \cdot 2^{2n} + (2^{2n} - 1)$$

Conclusion,

$$2^{2(n+1)} - 1 = 3 \cdot 2^{2n} + (2^{2n} - 1)$$

$$\begin{cases} 3 \cdot 2^{2n} \ is\ a\ multiple\ of\ 3, \ \ because\ there\ is\ the\ factor\ 3 \\ 2^{2n} - 1\ is\ a\ multiple\ of\ 3, \ \ by\ the\ inductive\ hypothesis \end{cases} \rightarrow$$

$$If\ each\ term\ is\ divisible\ by\ 3, then\ the\ sum\ is\ divisible\ by\ 3$$

$$2^{2(n+1)} - 1 = 3 \cdot 2^{2n} + (2^{2n} - 1)\ is\ a\ multiple\ of\ 3$$

Q.E.D.