ANSWER on Question #76084 – Math – Differential Equations

QUESTION

Solve the partial differential equation

$$(D + D' - 1)(D + 2D' - 3)z = 4 + 3x + 6y$$

SOLUTION

Let us use some known facts.

Let the given differential equation be

$$F(D, D') = f(x, y).$$

Factorize $F(D, D')$ into linear factors. Then use the following results:

Rule I. Corresponding to each non-repeated factor $(bD - aD' - c)$, the part of C.F. is taken as

$$e^{\left(\frac{cx}{b}\right)}\varphi(by + ax),\ \ if\ b \neq 0$$

We now have three particular cases of Rule I:

Rule IA. Take $c = 0$ in Rule I. Hence corresponding to each linear factor $(bD - aD')$, the part of C.F. is

$$\varphi(by + ax),\ \ if\ b \neq 0.$$

Rule IB. Take $a = 0$ in Rule I. Hence corresponding to each linear factor $(bD - c)$, the part of C.F. is

$$e^{\left(\frac{cx}{b}\right)}\varphi(by),\ \ if\ b \neq 0.$$

Rule IC. Take $a = c = 0$ and $b = 1$ in Rule I. Hence corresponding to each linear factor $(1 \cdot D)$, the part of C.F. is

$$\varphi(y).$$

In our case,

$$(D+D'-1)(D+2D'-3)z = 4+3x+6y \rightarrow z(x,y) = C.F.+P.I.$$

1 STEP: Let find C.F.

$$\begin{cases} (D+D'-1)z \\ (bD-aD'-c)z \end{cases} \rightarrow \begin{cases} b=1 \\ a=-1 \\ c=1 \end{cases} \rightarrow (C.F.)_1 = e^{\left(\frac{1\cdot x}{1}\right)} \cdot \varphi_1(1\cdot y + (-1)\cdot x) \rightarrow$$

$$\boxed{(C.F.)_1 = e^x \cdot \varphi_1(y-x), where\ \varphi_1\ is\ arbitrary\ function}$$

$$\begin{cases} (D+2D'-3)z \\ (bD-aD'-c)z \end{cases} \rightarrow \begin{cases} b=1 \\ a=-2 \\ c=3 \end{cases} \rightarrow (C.F.)_2 = e^{\left(\frac{3\cdot x}{1}\right)} \cdot \varphi_2(1\cdot y + (-2)\cdot x) \rightarrow$$

$$\boxed{(C.F.)_2 = e^{3x} \cdot \varphi_2(y-2x), where\ \varphi_2\ is\ arbitrary\ function}$$

Then,

$$C.F. = (C.F.)_1 + (C.F.)_2 \rightarrow \boxed{C.F. = e^x \cdot \varphi_1(y-x) + e^{3x} \cdot \varphi_2(y-2x)}$$

2 STEP: Let find P.I.

$$P.I. = \frac{1}{(D+D'-1)(D+2D'-3)}(4+3x+6y) =$$

$$= \frac{1}{(-1)\cdot(1-[D+D'])\cdot(-3)\cdot\left(1-\left[\frac{D+2D'}{3}\right]\right)}(4+3x+6y) =$$

$$= \frac{1}{3}\cdot(1-[D+D'])^{-1}\cdot\left(1-\left[\frac{D+2D'}{3}\right]\right)^{-1}(4+3x+6y) =$$

$$= \left[\frac{1}{1-x} = 1+x+x^2+x^3+x^4+\cdots, |x|<1\right] =$$

$$= \frac{1}{3}\cdot(1+[D+D']+[D+D']^2+\cdots)\cdot\left(1+\left[\frac{D+2D'}{3}\right]+\left[\frac{D+2D'}{3}\right]^2+\cdots\right)(4+3x+6y)$$

$$= \frac{1}{3} \cdot \left(1 + [D + D'] + \left[\frac{D + 2D'}{3}\right] + \cdots\right)(4 + 3x + 6y) =$$

$$= \frac{1}{3} \cdot \left(1 + D + D' + \frac{D}{3} + \frac{2D'}{3}\right)(4 + 3x + 6y) = \frac{1}{3}\left(1 + \frac{4D}{3} + \frac{5D'}{3}\right)(4 + 3x + 6y) =$$

$$= \frac{1}{3} \cdot \left(4 + 3x + 6y + \frac{4}{3} \cdot \frac{\partial}{\partial x}(4 + 3x + 6y) + \frac{5}{3} \cdot \frac{\partial}{\partial y}(4 + 3x + 6y)\right) =$$

$$= \frac{1}{3} \cdot \left(4 + 3x + 6y + \frac{4}{3} \cdot 3 + \frac{5}{3} \cdot 6\right) = \frac{1}{3} \cdot (4 + 3x + 6y + 4 + 10) =$$

$$= \frac{1}{3} \cdot (18 + 3x + 6y) = 6 + x + 2y$$

Then,

$$\boxed{P.I. = 6 + x + 2y}$$

Conclusion,

$$z(x, y) = C.F. + P.I. = e^{x} \cdot \varphi_1(y - x) + e^{3x} \cdot \varphi_2(y - 2x) + 6 + x + 2y$$

$$\boxed{\begin{cases} z(x, y) = e^{x} \cdot \varphi_1(y - x) + e^{3x} \cdot \varphi_2(y - 2x) + 6 + x + 2y \\ \quad where\ \varphi_1 and\ \varphi_2\ are\ arbitrary\ functions \end{cases}}$$

ANSWER:

$$\begin{cases} z(x, y) = e^{x} \cdot \varphi_1(y - x) + e^{3x} \cdot \varphi_2(y - 2x) + 6 + x + 2y \\ \quad where\ \varphi_1 and\ \varphi_2\ are\ arbitrary\ functions \end{cases}$$