Answer on Question #75930 – Math – Linear Algebra

Question

Given: $T:\mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation defined by

$$T(x,y,z)=(x+y,y,2x-2y+2z)$$

To prove or disprove: $(x-1)^2(x-2)$ is the characteristic polynomial and find minimal polynomial.

Solution

Consider the map $T:\mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(x,y,z)=(x+y,y,2x-2y+2z)$$

$\therefore$ Matrix representation of T with respect to standard basis $\{e_1,e_2,e_3\}$ is as follows

$$T(1,0,0)=(1,0,2)$$

$$T(0,1,0)=(1,1,-2)$$

$$T(0,0,1)=(0,0,2)$$

$\Rightarrow$ Matrix representation $[T]=A=\begin{bmatrix}1 & 0 & 2\\ 1 & 1 & -2\\ 0 & 0 & 2\end{bmatrix}$

Now we will check whether $(x-1)^2(x-2)$ is a characteristic polynomial.

So, replace x by A and 1 by I, we get

$$(A-I)^2(A-2I)$$

$$\Rightarrow \begin{bmatrix}0 & 0 & 2\\ 1 & 0 & -2\\ 0 & 0 & 1\end{bmatrix}^2\begin{bmatrix}-1 & 0 & 2\\ 1 & -1 & -2\\ 0 & 0 & 0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}0 & 0 & 2\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-1 & 0 & 2\\ 1 & -1 & -2\\ 0 & 0 & 0\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, *T* satisfies the polynomial.

To find the minimal polynomial, first we will find the polynomial with a linear factor of characteristic polynomial.

So, the product of linear factors is $(x-1)(x-2)$

Replace x by A and 1 by I, we get

$$(A-I)(A-2I)$$

$$\Rightarrow \quad \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 2 \\ 1 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, the minimal polynomial is same as the characteristic polynomial.

$$\therefore \quad m(x) = (x-1)^2(x-2)$$