Answer on Question #75698 – Math – Calculus

Question

Find the length of the curve $y = \ln\frac{e^x-1}{e^x+1}$ from $x = 1$ to $x = 2$.

Solution

We`ll use formula $L = \int_a^b \sqrt{(1 + (y')^2)}dx$

$$a = 1, b = 2$$

$$y' = \left(\ln\frac{e^x - 1}{e^x + 1}\right)' = \frac{e^x + 1}{e^x - 1}\cdot\left(\frac{e^x - 1}{e^x + 1}\right)' = \frac{e^x + 1}{e^x - 1}\cdot\frac{e^x(e^x + 1) - e^x(e^x - 1)}{(e^x + 1)^2} =$$

$$= \frac{e^x + 1}{e^x - 1}\cdot\frac{2e^x}{(e^x + 1)^2} = \frac{2e^x}{e^{2x} - 1}$$

$$1 + (y')^2 = 1 + \frac{4e^{2x}}{(e^{2x} - 1)^2} = \frac{4e^{2x} + (e^{2x} - 2)^2}{(e^{2x} - 1)^2} = \frac{4e^{2x} + e^{4x} - 2e^{2x} + 1}{(e^{2x} - 1)^2} =$$

$$= \frac{e^{4x} + 2e^{2x} + 1}{(e^{2x} - 1)^2} = \frac{(e^{2x} + 1)^2}{(e^{2x} - 1)^2}$$

$$\sqrt{1 + (y')^2} = \sqrt{\frac{(e^{2x} + 1)^2}{(e^{2x} - 1)^2}} = \frac{e^{2x} + 1}{e^{2x} - 1}$$

$$(y')^2 = \frac{4e^{2x}}{(e^{2x} - 1)^2}$$

$$L = \int_1^2 \frac{e^{2x} + 1}{e^{2x} - 1}dx = \left|\begin{array}{c} e^{2x} = t \\ x = \frac{1}{2}\ln t \\ dx = \frac{1}{2t}dt \\ x = 1 \Rightarrow t = e^2 \\ x = 2 \Rightarrow t = e^4 \end{array}\right| = \frac{1}{2}\int_{e^2}^{e^4} \frac{t + 1}{(t - 1)t}dt =$$

$$= \frac{1}{2}\int_{e^2}^{e^4} \frac{t}{(t - 1)t}dt + \frac{1}{2}\int_{e^2}^{e^4} \frac{1}{(t - 1)t}dt = \frac{1}{2}\int_{e^2}^{e^4} \frac{1}{t - 1}dt + \frac{1}{2}\int_{e^2}^{e^4} \frac{1 + t - t}{(t - 1)t}dt$$

$$= \frac{1}{2}\ln(t - 1)\Big|_{e^2}^{e^4} + \frac{1}{2}\int_{e^2}^{e^4} \left(-\frac{1}{t} + \frac{1}{t - 1}\right)dt$$

$$= \frac{1}{2}\ln(t - 1)\Big|_{e^2}^{e^4} - \frac{1}{2}\int_{e^2}^{e^4} \frac{1}{t}dt + \frac{1}{2}\int_{e^2}^{e^4} \frac{1}{t - 1}dt$$

$$= \frac{1}{2}\ln(t - 1)\Big|_{e^2}^{e^4} - \frac{1}{2}\ln t\Big|_{e^2}^{e^4} + \frac{1}{2}\ln(t - 1)\Big|_{e^2}^{e^4} = \ln(t - 1)\Big|_{e^2}^{e^4} - \frac{1}{2}(\ln e^4 - \ln e^2)$$

$$= \ln(e^4 - 1) - \ln(e^2 - 1) - \frac{1}{2}\ln\frac{e^4}{e^2} = \ln\frac{e^4 - 1}{e^2 - 1} - \frac{1}{2}\ln e^2$$

$$= \ln\frac{(e^2 + 1)(e^2 - 1)}{e^2 - 1} - \frac{1}{2}\cdot 2 = \ln(e^2 + 1) - 1 \approx 1.1269$$

Answer:

$L = \ln(e^2 + 1) - 1 \approx 1.1269$