Answer on Question #75616 – Math – Linear Algebra

Question

Determine all values of the constant a for which the following system has **(a)** no solution, **(b)** an infinite number of solutions, and **(c)** a unique solution.

$$ax_1 + x_2 + x_3 = 1,$$
$$x_1 + ax_2 + x_3 = 1,$$
$$x_1 + x_2 + ax_3 = 1.$$

Solution

The coefficient matrix

$$A = \begin{pmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{pmatrix}$$

Find

$$detA = \begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} = a\begin{vmatrix} a & 1 \\ 1 & a \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & a \end{vmatrix} + \begin{vmatrix} 1 & a \\ 1 & 1 \end{vmatrix} =$$

$$= a(a^2 - a) - (a - 1) + (1 - a) = (a - 1)(a(a + 1) - 2) =$$

$$= (a - 1)^2(a + 2)$$

The linear system has a unique solution iff $detA \neq 0$

$$(a - 1)^2(a + 2) \neq 0$$
$$a \neq -2, a \neq 1$$

If $a = 1$

$$x_1 + x_2 + x_3 = 1,$$
$$x_1 + x_2 + x_3 = 1,$$
$$x_1 + x_2 + x_3 = 1.$$

There is one equation for three variables.

The system has an infinite number of solutions.

If $a = -2$

$$-2x_1 + x_2 + x_3 = 1,$$
$$x_1 - 2x_2 + x_3 = 1,$$
$$x_1 + x_2 - 2x_3 = 1.$$

Add three equations

$$(-2x_1 + x_2 + x_3) + (x_1 - 2x_2 + x_3) + (x_1 + x_2 - 2x_3) = 1 + 1 + 1$$

We have

$$0 = 3, False$$

The system is inconsistent. Therefore, the system has no solutions.

Answer:

(a) If $a = -2$, the system has no solution.

(b) If $a = 1$, the system has an infinite number of solutions.

(c) If $a \neq -2, a \neq 1$ the system has a unique solution.