Answer on Question #75363 – Math – Quantitative Methods

Question

Write down only one application of Runge Kutta Method to solve some real world problem.

Solution

The rate of change of the temperature dT(t)/dt, is (by Newton's Law of Cooling) proportional to the difference between the temperature of the object T(t) and the ambient temperature Ta. This means that:

$$\frac{dT(t)}{dt} = -r(T(t) - Ta), \qquad r - \textit{positive constant characteristic of the system}$$

The analytical solution of this differential equation:

$$T(t) = Ta + C \cdot e^{-rt}$$

Let for t=0 T(0)=90 and r=0.1 and Ta=30 then:

$$\frac{dT(t)}{dt} = -0.1(T(t) - 30)\ with\ IVP: T(0) = 90 \qquad (1)$$

And exact solution:

$$T(t) = 30 + 60 \cdot e^{-0.1 \cdot t}$$

The Runge-Kutta-Fehlberg method is single-step method. This method has a procedure to determine whether the correct step size h is used. Each step requires the use of the following six values:

$$k_1 = h \cdot f(x_k, y_k)$$

$$k_2 = h \cdot f\left(x_k + \frac{1}{4}h, y_k + \frac{1}{4}k_1\right)$$

$$k_3 = h \cdot f\left(x_k + \frac{3}{8}h, y_k + \frac{3}{32}k_1 + \frac{9}{32}k_2\right)$$

$$k_4 = h \cdot f\left(x_k + \frac{12}{13}h, y_k + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3\right)$$

$$k_5 = h \cdot f\left(x_k + h, y_k + \frac{439}{216}k_1 - 8k_2 + \frac{3680}{513}k_3 - \frac{845}{4104}k_4\right)$$

$$k_6 = h \cdot f\left(x_k + \frac{1}{2}h, y_k - \frac{8}{27}k_1 + 2k_2 - \frac{3544}{2565}k_3 + \frac{1859}{4104}k_4 - \frac{11}{40}k_5\right)$$

Then we calculate the approximation to the solution with the help of the method of the fourth order:

$$w_{k+1} = y_k + \frac{25}{216}k_1 + \frac{1408}{2565}k_3 + \frac{2197}{4101}k_4 - \frac{1}{5}k_5, \qquad Error = O(h^4)$$

And the approximation to the solution with the help of the method of the 5th order:

$$y_{k+1} = y_k + \frac{16}{135}k_1 + \frac{6656}{12825}k_3 + \frac{28561}{56430}k_4 - \frac{9}{50}k_5 + \frac{2}{55}k_6, \qquad Error = O(h^5)$$

At each step, two different approximations for the solution are made and compared. The optimal step size is $(\delta \cdot h)$.

$$R = \frac{1}{h}|w_{k+1} - y_{k+1}|, \qquad \delta = \left(\frac{\varepsilon}{2R}\right)^{\frac{1}{4}} \quad \varepsilon - \text{specified accuracy}$$

Problem solving and error analysis:

$D(t,T) := -0.1\cdot(T-30) \qquad y_0 := 90$

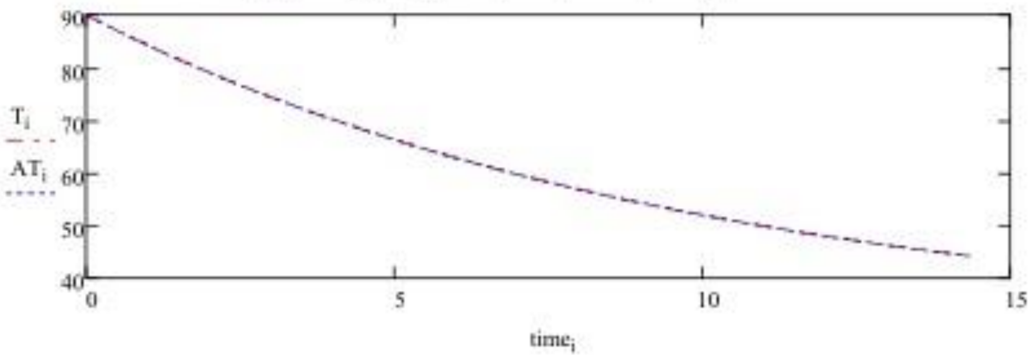
D(t,T) - 1st order ODE with IVP

$$\text{Runge45}(y0, D) := \left| \begin{array}{l} \left(x_0 \leftarrow 0 \quad y_0 \leftarrow y0 \quad h \leftarrow 10^{-6}\right) \\ \text{for } i \in 0..100 \\ \quad \left| \begin{array}{l} x_{i+1} \leftarrow x_i + h \\ k1 \leftarrow D(x_i, y_i) \\ k2 \leftarrow D\left(x_i + \frac{h}{4}, y_i + \frac{h\cdot k1}{4}\right) \\ k3 \leftarrow D\left(x_i + \frac{3\cdot h}{8}, y_i + \frac{h\cdot 3\cdot k1}{32} + \frac{h\cdot 9\cdot k2}{32}\right) \\ k4 \leftarrow D\left(x_i + \frac{12\cdot h}{13}, y_i + \frac{h\cdot 1932\cdot k1}{2197} - \frac{h\cdot 7200\cdot k2}{2197} + \frac{h\cdot 7296\cdot k3}{2197}\right) \\ k5 \leftarrow D\left(x_i + h, y_i + \frac{h\cdot 439\cdot k1}{216} - h\cdot 8\cdot k2 + \frac{h\cdot 3680\cdot k3}{513} - \frac{h\cdot 845\cdot k4}{4104}\right) \\ k6 \leftarrow D\left(x_i + \frac{h}{2}, y_i - \frac{h\cdot 8\cdot k1}{27} + h\cdot 2\cdot k2 - \frac{h\cdot 3544\cdot k3}{2565} + \frac{h\cdot 1859\cdot k4}{4104} - \frac{h\cdot 11\cdot k4}{40}\right) \\ w_{i+1} \leftarrow y_i + h\cdot\left(\frac{25\cdot k1}{216} + \frac{1408\cdot k3}{2565} + \frac{2197\cdot k4}{4101} - \frac{k5}{5}\right) \\ z_{i+1} \leftarrow y_i + h\cdot\left(\frac{16\cdot k1}{135} + \frac{6656\cdot k3}{12825} + \frac{28561\cdot k4}{56430} - \frac{9\cdot k5}{50} + \frac{2\cdot k6}{55}\right) \\ y_{i+1} \leftarrow z_{i+1} \\ h \leftarrow \left(\frac{h\cdot 10^{-6}}{2\cdot\left|z_{i+1} - w_{i+1}\right|}\right)^{\frac{1}{4}} \end{array} \right. \\ \text{return augment}(x, y) \end{array} \right.$$

Error calculation.
T - exact value
AT - approximation

$Z := \text{Runge45}(y_0, D) \qquad fT(t) := 30 + 60\cdot e^{-0.1\cdot t}$

$i := 0..\text{rows}(Z) - 1 \qquad time_i := Z_{i,0} \qquad T_i := fT(time_i) \qquad AT_i := Z_{i,1} \qquad |T - AT| = 0.00002566293$



The classical fourth-order Runge-Kutta method for computations with a constant integration step requires the calculation of four coefficients:

$$k_1 = f(x_k, y_k)$$

$$k_2 = f\left(x_k + \frac{1}{2}h, y_k + \frac{1}{2}k_1\right)$$

$$k_3 = f\left(x_k + \frac{1}{2}h, y_k + \frac{1}{2}k_2\right)$$

$$k_4 = f(x_k + h, y_k + hk_3)$$

Then we calculate the approximation to the solution with the help of the method of the fourth order:

$$y_{k+1} = y_k + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Problem solving and error analysis:

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Runge4(y0,step,D) := | (x_0 ← 0  y_0 ← y0  h ← step)
                     | for i ∈ 0..100
                     |   | x_{i+1} ← x_i + h
                     |   | k1 ← D(x_i, y_i)
                     |   | k2 ← D(x_i + h/2, y_i + h·k1/2)
                     |   | k3 ← D(x_i + h/2, y_i + h·k2/2)
                     |   | k4 ← D(x_i + h, y_i + h·k3)
                     |   | y_{i+1} ← y_i + h·((k1 + 2·k2 + 2·k3 + k4)/6)
                     | return augment(x,y)
```

Error calculation.
T - exact value
AT - approximation

$Z := \text{Runge4}(y_0, 0.15, D)$

$i := 0..\text{rows}(Z) - 1 \qquad \text{time}_i := Z_{i,0} \qquad T_i := fT(\text{time}_i) \qquad AT_i := Z_{i,1} \qquad |T - AT| = 0.00000008016$