

ANSWER on Question #74999 – Math – Calculus

If

$$y(x) = e^{m \cdot \operatorname{atan} x}$$

Show that

$$(1 + x^2)y^{(n+1)} + (2nx - 1)y^{(n)} + n(n - 1)y^{(n-1)} = 0, \quad \forall n \geq 1$$

SOLUTION

Let us agree on the notation

$$y^{(n)} \equiv \frac{d^n y}{dx^n}$$

$$\frac{dy}{dx} \equiv y'$$

$$\frac{d}{dx}(y^{(n)}) = y^{(n+1)}$$

We recall some necessary rules for taking derivatives

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$$

$$\frac{d}{dx}(f(x) \cdot g(x)) = \frac{d}{dx}(f(x)) \cdot g(x) + f(x) \cdot \frac{d}{dx}(g(x))$$

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$$

$$\frac{d}{dx}(c \cdot f(x)) = c \cdot \frac{d}{dx}(f(x))$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(\operatorname{atan}(x)) = \frac{1}{1 + x^2}$$

(More information: <https://en.wikipedia.org/wiki/Derivative>)

In our case, we derive this recurrence relation

$$y^{(0)} \equiv y = e^{m \cdot \arctan x} \rightarrow \frac{d}{dx} \times \left| y = e^{m \cdot \arctan x} \rightarrow \frac{dy}{dx} = \frac{d}{dx} (e^{m \cdot \arctan x}) \rightarrow$$

$$y^{(1)} = e^{m \cdot \arctan x} \cdot \frac{m}{1+x^2} \left| \times (1+x^2) \rightarrow (1+x^2)y^{(1)} = \underbrace{me^{m \cdot \arctan x}}_{y^{(0)}} \rightarrow \boxed{(1+x^2)y^{(1)} = my^{(0)}}$$

$$\boxed{(1+x^2)y^{(1)} - my^{(0)} = 0}$$

$$\frac{d}{dx} \times \left| (1+x^2)y^{(1)} - my^{(0)} = 0 \rightarrow \frac{d}{dx} \left((1+x^2)y^{(1)} \right) - \frac{d}{dx} (my^{(0)}) = 0$$

$$2xy^{(1)} + (1+x^2)y^{(2)} - my^{(1)} = 0 \rightarrow \boxed{(1+x^2)y^{(2)} + (2x-m)y^{(1)} = 0}$$

$$(1+x^2)y^{(2)} + (2x-m)y^{(1)} = 0 \leftrightarrow$$

$$(1+x^2)y^{(1+1)} + (2 \cdot 1 \cdot x - m)y^{(1)} + 1 \cdot (1-1) \cdot y^{(0)} = 0$$

$$\frac{d}{dx} \times \left| (1+x^2)y^{(2)} + (2x-m)y^{(1)} = 0 = 0 \rightarrow \frac{d}{dx} \left((1+x^2)y^{(2)} \right) + \frac{d}{dx} \left((2x-m)y^{(1)} \right) = 0$$

$$2xy^{(2)} + (1+x^2)y^{(3)} + 2y^{(1)} + (2x-m)y^{(2)} = 0 \rightarrow$$

$$(1+x^2)y^{(3)} + (2x+2x-m)y^{(2)} + 2y^{(1)} = 0 \rightarrow$$

$$\boxed{(1+x^2)y^{(3)} + (4x-m)y^{(2)} + 2y^{(1)} = 0}$$

$$(1+x^2)y^{(2+1)} + (2 \cdot 2 \cdot x - m)y^{(2)} + 2 \cdot (2-1)y^{(1)} = 0$$

$$\frac{d}{dx} \times \left| (1+x^2)y^{(3)} + (4x-m)y^{(2)} + 2y^{(1)} = 0 \rightarrow$$

$$\frac{d}{dx} \left((1+x^2)y^{(3)} \right) + \frac{d}{dx} \left((4x-m)y^{(2)} \right) + \frac{d}{dx} (2y^{(1)}) = 0 \rightarrow$$

$$2xy^{(3)} + (1+x^2)y^{(4)} + 4y^{(2)} + (4x-m)y^{(3)} + 2y^{(2)} = 0 \rightarrow$$

$$(1+x^2)y^{(4)} + (4x+2x-m)y^{(3)} + 6y^{(2)} = 0 \rightarrow$$

$$\boxed{(1+x^2)y^{(4)} + (6x-m)y^{(3)} + 6y^{(2)} = 0}$$

$$(1 + x^2)y^{(3+1)} + (2 \cdot 3 \cdot x - m)y^{(3)} + 3 \cdot (3 - 1)y^{(2)} = 0$$

Conclusion,

$$(1 + x^2)y^{(n+1)} + (2nx - m)y^{(n)} + n(n - 1)y^{(n-1)} = 0$$

It remains to prove this relation.

We use the method of mathematical induction

1 STEP : Basis of induction

For $n = 2$

$$(1 + x^2)y^{(2+1)} + (2 \cdot 2 \cdot x - m)y^{(2)} + 2 \cdot (2 - 1)y^{(2-1)} = 0 \rightarrow$$

$$(1 + x^2)y^{(3)} + (4x - m)y^{(2)} + 2y^{(1)} = 0$$

We have already obtained this formula when we derive this recurrence relation

2 STEP : Inductive hypothesis

Suppose that the formula holds $\forall k: 1 \leq k \leq n$

$$(1 + x^2)y^{(k+1)} + (2kx - m)y^{(k)} + k(k - 1)y^{(k-1)} = 0$$

3 STEP : The inductive step

It is necessary to prove that the formula is true for $k = n + 1$

$$(1 + x^2)y^{(n+2)} + (2(n + 1)x - m)y^{(n+1)} + n(n + 1)y^{(n)} = 0$$

In our case,

$$\frac{d}{dx} \times \left| (1 + x^2)y^{(n+1)} + (2nx - m)y^{(n)} + n(n - 1)y^{(n-1)} = 0 \rightarrow \right.$$

$$\frac{d}{dx} \left((1 + x^2)y^{(n+1)} \right) + \frac{d}{dx} \left((2nx - m)y^{(n)} \right) + \frac{d}{dx} \left(n(n - 1)y^{(n-1)} \right) = 0 \rightarrow$$

$$2xy^{(n+1)} + (1 + x^2)y^{(n+2)} + 2n \cdot y^{(n)} + (2nx - m)y^{(n+1)} + n(n - 1)y^{(n)} = 0 \rightarrow$$

$$(1 + x^2)y^{(n+2)} + (2x + 2nx - m)y^{(n+1)} + (2n + n(n - 1))y^{(n)} = 0 \rightarrow$$

$$(1 + x^2)y^{(n+2)} + (2(n + 1)x - m)y^{(n+1)} + n(n + 1)y^{(n)} = 0. \quad Q.E.D$$

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