Question #74942 - Math – Quantitative Methods

use modified euler 's method to find the approximate solution of IVP y' =2xy, y(1)=1, at x=1.5 with h=0.1. if the exact solution is y(x) =ex2-1, find the error.

Solution: for the numerical solution of the differential equation:

$$y' = f(x,y), \qquad f(x,y) = 2 \cdot x \cdot y$$

use the explicit midpoint method also known as the modified Euler method:

$$y_{n+1} = y_n + h \cdot f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} \cdot f(x_n, y_n)\right), \qquad h = 0.1, \qquad x_0 = 1, \qquad y_0 = 1$$

to find the approximate solution of IVP at x=1.5, it takes 5 steps:

$$x_1 = x_0 + h = 1.1, \;\; y_1 = y_0 + h \cdot f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} \cdot f(x_0, y_0)\right)$$
$$= 1 + 0.1 \cdot f\big(1.05, 1 + 0.05 \cdot f(1,1)\big) = 1 + 0.1 \cdot f(1.05, 1.1) = 1.2310$$

$$x_2 = x_1 + h = 1.2, \;\; y_2 = y_1 + h \cdot f\left(x_1 + \frac{h}{2}, y_1 + \frac{h}{2} \cdot f(x_1, y_1)\right)$$
$$= 1.231 + 0.1 \cdot f\big(1.15, 1.231 + 0.05 \cdot f(1.1, 1.231)\big)$$
$$= 1.231 + 0.1 \cdot f(1.15, 1.3664) = 1.5453$$

$$x_3 = x_2 + h = 1.3, \;\; y_3 = y_2 + h \cdot f\left(x_2 + \frac{h}{2}, y_2 + \frac{h}{2} \cdot f(x_2, y_2)\right)$$
$$= 1.5453 + 0.1 \cdot f\big(1.25, 1.5453 + 0.05 \cdot f(1.2, 1.5453)\big)$$
$$= 1.5453 + 0.1 \cdot f(1.25, 1.7307) = 1.97795$$

$$x_4 = x_3 + h = 1.4, \;\; y_4 = y_3 + h \cdot f\left(x_3 + \frac{h}{2}, y_3 + \frac{h}{2} \cdot f(x_3, y_3)\right)$$
$$= 1.97795 + 0.1 \cdot f\big(1.35, 1.97795 + 0.05 \cdot f(1.3, 1.97795)\big)$$
$$= 1.97795 + 0.1 \cdot f(1.35, 2.2351) = 2.5814$$

$$x_5 = x_4 + h = 1.5, \;\; y_5 = y_4 + h \cdot f\left(x_4 + \frac{h}{2}, y_4 + \frac{h}{2} \cdot f(x_4, y_4)\right)$$
$$= 2.5814 + 0.1 \cdot f\big(1.45, 2.5814 + 0.05 \cdot f(1.4, 2.5814)\big)$$
$$= 2.5814 + 0.1 \cdot f(1.45, 2.9428) = 3.4348$$

exact solution:

$$y(x) = e^{x^2-1}, \; y(1.5) = 3.4903 \quad error = 3.4903 - 3.4348 = 0.0555 \;\; (\sim 1.6\%)$$

Answer: $y_5 = 3.4348, \;\; error = 0.0555 \;\; (\sim 1.6\%)$