

Answer on Question #74941, Math / Calculus

Solve the following ODE using power series method.

$$(x + 2)y'' + xy' - y = 0$$

Solution

Let assume the solution in form $y = \sum_{n=0}^{\infty} a_n x^n$, exists

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute

$$(x + 2) \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} 2n(n-1) a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1} + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{N=0}^{\infty} 2(N+2)(N+1) a_{N+2} x^N + \sum_{N=1}^{\infty} N(N+1) a_{N+1} x^N + \sum_{N=1}^{\infty} N a_N x^N - \sum_{N=0}^{\infty} a_N x^N = 0$$

$$2(0+2)(0+1) a_{0+2} x^0 + \sum_{N=1}^{\infty} 2(N+2)(N+1) a_{N+2} x^N + \\ + \sum_{N=1}^{\infty} N(N+1) a_{N+1} x^N + \sum_{N=1}^{\infty} N a_N x^N - c_0 x^0 - \sum_{N=1}^{\infty} a_N x^N = 0$$

$$4a_2 - a_0 = 0$$

$$\sum_{N=1}^{\infty} x^N [2(N+2)(N+1) a_{N+2} + N(N+1) a_{N+1} + (N-1) a_N] = 0$$

$$2(N+2)(N+1) a_{N+2} + N(N+1) a_{N+1} + (N-1) a_N = 0, N \geq 1$$

$$a_{N+2} = -\frac{N(N+1) a_{N+1} + (N-1) a_N}{2(N+2)(N+1)}$$

Plug in $a_0 = 1, a_1 = 0$ to find the first solution $y_1(x)$

$$4a_2 - 1 = 0 \Rightarrow a_2 = \frac{1}{4}$$

$$a_3 = -\frac{1(1+1)\left(\frac{1}{4}\right) + (1-1)(0)}{2(1+2)(1+1)} = -\frac{1}{24}$$

$$a_4 = -\frac{2(2+1)\left(-\frac{1}{24}\right) + (2-1)\left(\frac{1}{4}\right)}{2(2+2)(2+1)} = 0$$

$$a_5 = -\frac{3(3+1)(0) + (3-1)\left(-\frac{1}{24}\right)}{2(3+2)(3+1)} = \frac{1}{480}$$

$$a_6 = -\frac{4(4+1)\left(\frac{1}{480}\right) + (4-1)(0)}{4(4+2)(4+1)} = -\frac{1}{2880}$$

$$y_1(x) = 1 + \frac{1}{4}x^2 - \frac{1}{24}x^3 + \frac{1}{480}x^5 - \frac{1}{2880}x^6 + \dots$$

Plug in $a_0 = 0, a_1 = 1$ to find the second solution $y_2(x)$
 $4a_2 - a_0 = 0 \Rightarrow a_2 = 0$

$$a_3 = -\frac{1(1+1)(0) + (1-1)(1)}{2(1+2)(1+1)} = 0$$

$$a_4 = -\frac{2(2+1)(0) + (2-1)(0)}{2(2+2)(2+1)} = 0$$

$$a_5 = -\frac{3(3+1)(0) + (3-1)(0)}{2(3+2)(3+1)} = 0$$

$$a_6 = -\frac{4(4+1)(0) + (4-1)(0)}{4(4+2)(4+1)} = 0$$

$$y_2(x) = x$$

The general solution is

$$y = c_1y_1 + c_2x$$

Answer: $y = c_1y_1 + c_2x,$

$$y_1(x) = 1 + \frac{1}{4}x^2 - \frac{1}{24}x^3 + \frac{1}{480}x^5 - \frac{1}{2880}x^6 + \dots$$

$$y_2(x) = x$$

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