ANSWER on Question #74826 – Math – Calculus

If

$$y(x) = e^{\operatorname{atan} x}$$

Show that

$$(1 + x^2)y^{(n+1)} + (2nx - 1)y^{(n)} + n(n-1)y^{(n-1)} = 0, \quad \forall n \geq 1$$

SOLUTION

Let us agree on the notation

$$y^{(n)} \equiv \frac{d^n y}{dx^n} \qquad \frac{dy}{dx} \equiv y' \qquad \frac{d}{dx}\left(y^{(n)}\right) = y^{(n+1)}$$

We recall some necessary rules for taking derivatives

$$\frac{d}{dx}\big(f(x) + g(x)\big) = \frac{d}{dx}\big(f(x)\big) + \frac{d}{dx}\big(g(x)\big)$$

$$\frac{d}{dx}\big(f(x) \cdot g(x)\big) = \frac{d}{dx}\big(f(x)\big) \cdot g(x) + f(x) \cdot \frac{d}{dx}\big(g(x)\big)$$

$$\frac{d}{dx}\Big(f\big(g(x)\big)\Big) = f'\big(g(x)\big) \cdot g'(x)$$

$$\frac{d}{dx}\big(c \cdot f(x)\big) = c \cdot \frac{d}{dx}\big(f(x)\big)$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(\operatorname{atan}(x)) = \frac{1}{1 + x^2}$$

(More information: https://en.wikipedia.org/wiki/Derivative)

In our case, we derive this recurrence relation

$$y^{(0)} \equiv y = e^{\operatorname{atan} x} \rightarrow \frac{d}{dx} \times \Big| y = e^{\operatorname{atan} x} \rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(e^{\operatorname{atan} x}\right) \rightarrow$$

$$y^{(1)} = e^{\operatorname{atan} x} \cdot \frac{1}{1+x^2}\bigg| \times (1+x^2) \rightarrow (1+x^2)y^{(1)} = \underbrace{e^{\operatorname{atan} x}}_{y^{(0)}} \rightarrow \boxed{(1+x^2)y^{(1)} = y^{(0)}}$$

$$\boxed{(1+x^2)y^{(1)} - y^{(0)} = 0}$$

$$\frac{d}{dx}\times\bigg| (1+x^2)y^{(1)} - y^{(0)} = 0 \rightarrow \frac{d}{dx}\left((1+x^2)y^{(1)}\right) - \frac{d}{dx}\left(y^{(0)}\right) = 0$$

$$2xy^{(1)} + (1+x^2)y^{(2)} - y^{(1)} = 0 \rightarrow \boxed{(1+x^2)y^{(2)} + (2x-1)y^{(1)} = 0}$$

$$(1+x^2)y^{(2)} + (2x-1)y^{(1)} = 0 \leftrightarrow$$

$$(1+x^2)y^{(1+1)} + (2\cdot 1\cdot x - 1)y^{(1)} + 1\cdot(1-1)\cdot y^{(0)} = 0$$

$$\frac{d}{dx}\times\bigg| (1+x^2)y^{(2)} + (2x-1)y^{(1)} = 0 = 0 \rightarrow \frac{d}{dx}\left((1+x^2)y^{(2)}\right) + \frac{d}{dx}\left((2x-1)y^{(1)}\right) = 0$$

$$2xy^{(2)} + (1+x^2)y^{(3)} + 2y^{(1)} + (2x-1)y^{(2)} = 0 \rightarrow$$

$$(1+x^2)y^{(3)} + (2x+2x-1)y^{(2)} + 2y^{(1)} = 0 \rightarrow \boxed{(1+x^2)y^{(3)} + (4x-1)y^{(2)} + 2y^{(1)} = 0}$$

$$(1+x^2)y^{(2+1)} + (2\cdot 2\cdot x - 1)y^{(2)} + 2\cdot(2-1)y^{(1)} = 0$$

$$\frac{d}{dx}\times\bigg| (1+x^2)y^{(3)} + (4x-1)y^{(2)} + 2y^{(1)} = 0 \rightarrow$$

$$\frac{d}{dx}\left((1+x^2)y^{(3)}\right) + \frac{d}{dx}\left((4x-1)y^{(2)}\right) + \frac{d}{dx}\left(2y^{(1)}\right) = 0 \rightarrow$$

$$2xy^{(3)} + (1+x^2)y^{(4)} + 4y^{(2)} + (4x-1)y^{(3)} + 2y^{(2)} = 0 \rightarrow$$

$$(1+x^2)y^{(4)} + (4x+2x-1)y^{(3)} + 6y^{(2)} = 0 \rightarrow \boxed{(1+x^2)y^{(4)} + (6x-1)y^{(3)} + 6y^{(2)} = 0}$$

$$(1+x^2)y^{(3+1)} + (2\cdot 3\cdot x - 1)y^{(3)} + 3\cdot(3-1)y^{(2)} = 0$$

Conclusion,

$$(1+x^2)y^{(n+1)} + (2nx-1)y^{(n)} + n(n-1)y^{(n-1)} = 0$$

It remains to prove this relation.

We use the method of mathematical induction

1 STEP : Basis of induction

For $n = 2$

$$(1+x^2)y^{(2+1)} + (2\cdot 2\cdot x - 1)y^{(2)} + 2\cdot(2-1)y^{(2-1)} = 0 \rightarrow$$

$$(1+x^2)y^{(3)} + (4x-1)y^{(2)} + 2y^{(1)} = 0$$

We have already obtained this formula when we derive this recurrence relation

2 STEP : Inductive hypothesis

Suppose that the formula holds $\forall k: 1 \leq k \leq n$

$$(1+x^2)y^{(k+1)} + (2kx-1)y^{(k)} + k(k-1)y^{(k-1)} = 0$$

3 STEP : The inductive step

It is necessary to prove that the formula is true for $k = n+1$

$$(1+x^2)y^{(n+2)} + (2(n+1)x-1)y^{(n+1)} + n(n+1)y^{(n)} = 0$$

In our case,

$$\frac{d}{dx}\times\Big|\,(1+x^2)y^{(n+1)} + (2nx-1)y^{(n)} + n(n-1)y^{(n-1)} = 0 \rightarrow$$

$$\frac{d}{dx}\Big((1+x^2)y^{(n+1)}\Big) + \frac{d}{dx}\Big((2nx-1)y^{(n)}\Big) + \frac{d}{dx}\Big(n(n-1)y^{(n-1)}\Big) = 0 \rightarrow$$

$$2xy^{(n+1)} + (1+x^2)y^{(n+2)} + 2n\cdot y^{(n)} + (2nx-1)y^{(n+1)} + n(n-1)y^{(n)} = 0 \rightarrow$$

$$(1+x^2)y^{(n+2)} + (2x+2nx-1)y^{(n+1)} + \big(2n + n(n-1)\big)y^{(n)} = 0 \rightarrow$$

$$(1+x^2)y^{(n+2)} + (2(n+1)x-1)y^{(n+1)} + n(n+1)y^{(n)} = 0. \qquad Q.E.D$$