Question #74803 Math – Combinatorics – Number Theory

Use the fact that

$$nCk \equiv \frac{n!}{k!\,(n-k)!}$$

to express in factorials

1) The coefficient "u" of x^n in the expansion of $(1+x)^{2n}$

2) The coefficient "v" of x^n in the expansion of $(1+x)^{2n-1}$. Hence show that $u = 2v$.

SOLUTION

By the definition,

$$nCk \equiv \frac{n!}{k!\,(n-k)!}$$

(More information: https://en.wikipedia.org/wiki/Binomial_coefficient)

By the definition,

$$n! \equiv 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot n$$

$$n! = (n-1)! \cdot n$$

(More information: https://en.wikipedia.org/wiki/Factorial)

Without proof, we assume the validity of formula

$$(x+y)^n = \sum_{k=0}^{n} (nCk) \cdot x^{n-k} y^k$$

(More information: https://en.wikipedia.org/wiki/Binomial_coefficient)

In our case,

1) The coefficient "u" of x^n in the expansion of $(1+x)^{2n}$

$$(1+x)^{2n}=\sum_{k=0}^{2n}(2nCk)1^{2n-k}x^k \equiv \sum_{k=0}^{2n}(2nCk)x^k$$

Near the monomial x^n is the coefficient

$$u=2nCn=\frac{(2n)!}{n!\cdot(2n-n)!}\rightarrow \boxed{u=\frac{(2n)!}{(n!)^2}}$$

2) The coefficient "v" of x^n in the expansion of $(1+x)^{2n-1}$.

$$(1+x)^{2n-1}=\sum_{k=0}^{2n-1}([2n-1]Ck)1^{(2n-1)-k}x^k \equiv \sum_{k=0}^{2n-1}([2n-1]Ck)x^k$$

Near the monomial x^n is the coefficient

$$v=[2n-1]Cn=\frac{(2n-1)!}{n!\cdot(2n-1-n)!}\rightarrow \boxed{v=\frac{(2n-1)!}{n!\cdot(n-1)!}}$$

It remains to show that

$$u=2v\rightarrow\frac{u}{v}=2$$

In our case,

$$\frac{u}{v}=\frac{\frac{(2n)!}{(n!)^2}}{\frac{(2n-1)!}{n!\cdot(n-1)!}}=\frac{(2n)!}{(n!)^2}\cdot\frac{n!\cdot(n-1)!}{(2n-1)!}=\frac{(2n-1)!\cdot 2n}{n!\cdot n!}\cdot\frac{n!\cdot(n-1)!}{(2n-1)!}=$$

$$=\frac{(2n-1)!}{\underbrace{(2n-1)!}_{=1}}\cdot\frac{n!}{\underbrace{n!}_{=1}}\cdot\frac{2n\cdot(n-1)!}{n!}=1\cdot 1\cdot\frac{2n\cdot(n-1)!}{(n-1)!\cdot n}=\frac{(n-1)!}{\underbrace{(n-1)!}_{=1}}\cdot\frac{n}{\underbrace{n}_{=1}}\cdot\frac{2}{\underbrace{1}_{=2}}=2$$

Conclusion,

$$\boxed{\frac{u}{v} = 2 \rightarrow u = 2v}$$

ANSWER

$$u = \frac{(2n)!}{(n!)^2}$$

$$v = \frac{(2n-1)!}{n! \cdot (n-1)!}$$

$$u = 2v$$