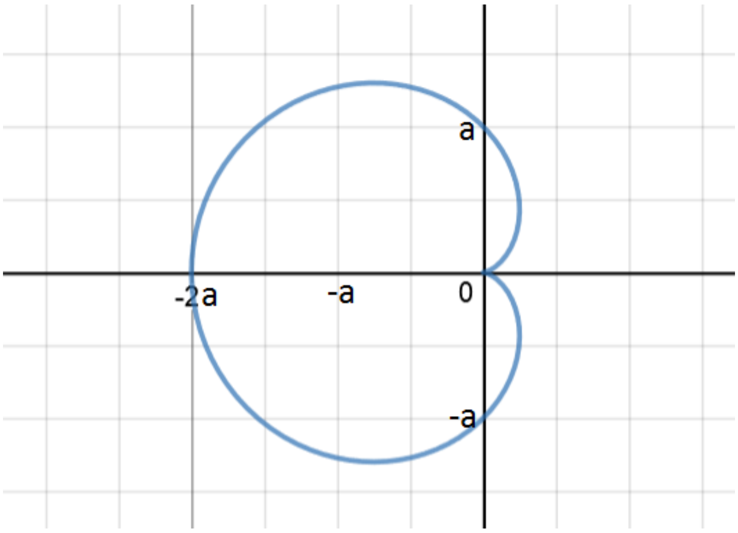
Solution for question #74733, Math / Calculus

Find the area enclosed by the curve $r = a(1 - \cos\theta)$.

Solution

The given curve has the following graph:



The formula for finding an area enclosed by a curve r in polar coordinates is $A = \frac{1}{2}\int_\alpha^\beta r^2 d\theta$.

For $0 \le \theta \le 2\pi$, $-1 \le \cos\theta \le 1 \Rightarrow 0 \le (1 - \cos\theta) \le 2$.

Therefore $r \ge 0$ for $0 \le \theta \le 2\pi$. Hence the area enclosed by the given curve is given by

$$A = \frac{1}{2}\int_0^{2\pi} \big(a(1-\cos\theta)\big)^2 d\theta = \frac{a^2}{2}\int_0^{2\pi} (1-\cos\theta)^2 d\theta = \frac{a^2}{2}\int_0^{2\pi} (\cos^2\theta - 2\cos\theta + 1) d\theta =$$

$$= \frac{a^2}{2}\int_0^{2\pi} \left(\frac{1+\cos 2\theta}{2} - 2\cos\theta + 1\right) d\theta = \frac{a^2}{2}\left(\int_0^{2\pi} \frac{\cos 2\theta}{4} d2\theta - \int_0^{2\pi} 2\cos\theta\, d\theta + \int_0^{2\pi} \frac{3}{2} d\theta\right) =$$

$$= \frac{a^2}{2}\left(\int_0^{2\pi} \frac{\cos 2\theta}{4} d2\theta - \int_0^{2\pi} 2\cos\theta\, d\theta + \int_0^{2\pi} \frac{3}{2} d\theta\right) = \frac{a^2}{2}\left(\frac{\sin 2\theta}{4} - 2\sin\theta + \frac{3\theta}{2}\right)\Bigg|_0^{2\pi} =$$

$$= \frac{a^2}{2}\left(\frac{\sin 4\pi}{4} - 2\sin 2\pi + \frac{6\pi}{2}\right) - \frac{a^2}{2}\left(\frac{\sin 0}{4} - 2\sin 0 + \frac{3*0}{2}\right) =$$

$$= \frac{a^2}{2}\left(\frac{0}{4} - 2*0 + 3\pi\right) - \frac{a^2}{2}\left(\frac{0}{4} - 2*0 + 0\right) = \frac{a^2}{2}(3\pi - 0) = \frac{3\pi a^2}{2}.$$

Answer

$A = \frac{3\pi a^2}{2}$.