Answer on Question #74602 – Math – Algebra

Question

Prove that

$$\frac{1}{2}(x+y+z) \leq \frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y} \text{ , for } x, y, z > 0$$

Solution

Theorem (Cauchy-Schwarz inequality)

Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be real numbers. Then we have

$$\left(\sum_{i=1}^{n} {a_i}^2\right)\left(\sum_{i=1}^{n} {b_i}^2\right) \geq \left(\sum_{i=1}^{n} a_i b_i\right)^2$$

Corollary

Let a, b, x, y be real numbers and $u, v > 0$. Then we have

$$\frac{a^2}{u} + \frac{b^2}{v} \geq \frac{(a+b)^2}{u+v}$$

Proof

The given inequality is equivalent to

$$v(u+v)a^2 + u(u+v)b^2 \geq uv(a+b)^2$$

$$a^2uv + a^2v^2 + b^2u^2 + b^2uv - a^2uv - 2abuv - b^2uv \geq 0$$

$$a^2v^2 + b^2u^2 - 2abuv \geq 0$$

$$(av - bu)^2 \geq 0,$$

which is clearly true.

Apply the inequality for $a = x, b = y, u = y + z, v = x + z, x, y, z > 0$

$$\frac{x^2}{y+z} + \frac{y^2}{x+z} \geq \frac{(x+y)^2}{x+y+2z}$$

Apply the inequality for $a = x + y, b = z, u = x + y + 2z, v = x + y, x, y, z > 0$

$$\frac{(x+y)^2}{x+y+2z} + \frac{z^2}{x+y} \geq \frac{(x+y+z)^2}{x+y+2z+x+y}$$

Hence

$$\frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y} \geq \frac{(x+y)^2}{x+y+2z} + \frac{z^2}{x+y} \geq \frac{(x+y+z)^2}{x+y+2z+x+y}$$

$$\frac{(x+y+z)^2}{x+y+2z+x+y} = \frac{(x+y+z)^2}{2(x+y+z)} = \frac{1}{2}(x+y+z)$$

Therefore, we prove that

$$\frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y} \geq \frac{1}{2}(x+y+z), \text{for } x, y, z > 0$$