Answer on Question #74601 – Math – Discrete Mathematics
Question

Show that

$$1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}\geq\sqrt{2}\sqrt{n-1}\text{ , for } n \text{ belongs to } N, n>1$$

Solution

For any $n\geq 2$, let P_n be the statement that

$$1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}\geq\sqrt{2}\sqrt{n-1}$$

Base case

The statement P_2 says that

$$1+\frac{1}{\sqrt{2}}\geq\sqrt{2}\sqrt{2-1},$$

$$1+\sqrt{2}+\frac{1}{2}\geq 2$$

$$\sqrt{2}\geq\frac{1}{2}$$

which is true.

Inductive case

Fix $k>2$, and suppose that P_k holds, that is,

$$1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{k}}\geq\sqrt{2}\sqrt{k-1}$$

It remains to show that P_{k+1} holds, that is, that

$$1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}}\geq\sqrt{2}\sqrt{k+1-1}$$

By P_k, $1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{k}}\geq\sqrt{2}\sqrt{k-1}$

$$1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}}\geq\sqrt{2}\sqrt{k-1}+\frac{1}{\sqrt{k+1}}$$

Let

$$\sqrt{2}\sqrt{k-1}+\frac{1}{\sqrt{k+1}}\geq\sqrt{2}\sqrt{k+1-1}$$

Then

$$2(k-1)+2\frac{\sqrt{2}\sqrt{k-1}}{\sqrt{k+1}}+\frac{1}{k+1}\geq 2k$$

Show that

$$2\frac{\sqrt{2}\sqrt{k-1}}{\sqrt{k+1}}\geq 2$$

$$\sqrt{2}\sqrt{k-1}\geq\sqrt{k+1}$$

$2k - 2 \geq k + 1$
$k \geq 3$
Hence

$$2\frac{\sqrt{2}\sqrt{k-1}}{\sqrt{k+1}} + \frac{1}{k+1} > 2\frac{\sqrt{2}\sqrt{k-1}}{\sqrt{k+1}} \geq 2, \text{for } k \geq 3$$

Therefore, P_{k+1} holds.
Thus by the principle of mathematical induction, for all $n \geq 1$, P_n holds

$$1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{2}\sqrt{n-1}, \qquad n \text{ belongs to } N, n > 1$$