ANSWER on Question #74168 – Math – Functional Analysis

A and B are subsets of $\mathbb{R}$ where $f : A \to B$ is a function. Prove that f is a bijection if and only if $f^{-1} : B \to A$ (f inverse from B to A) exists as a bijection.

SOLUTION

Let us recall several necessary definitions

Definition1. Let $f : A \to B$. We say that f is surjective if for all $b \in B$, there exists an

$a \in A$ such that $f(a) = b$.

Definition2. Let $f : A \to B$. We say that f is injective if whenever $f(a_1) = f(a_2)$ for some

$a_1, a_2 \in A$, then $a_1 = a_2$.

Definition3. We say that f is bijective if it is both injective and surjective.

Definition4. Let $f : A \to B$. A function $f^{-1} : B \to A$ is the inverse of f if

$f \circ f^{-1} = 1$ and $f^{-1} \circ f = 1$.

Let us prove two theorems:

Theorem 1. Let $f : A \to B$ be bijective. Then f has an inverse.

Proof. Let $f : A \to B$ be bijective. We will define a function $f^{-1} : B \to A$ as follows.

Let $b \in B$. Since f is surjective, there exists $a \in A$ such that $f(a) = b$. Let $f^{-1}(b) = a$. Since f is injective, this a is unique, so f^{-1} is well-defined.

Now we much check that f^{-1} is the inverse of f.

First we will show that $f^{-1} \circ f = 1$. Let $a \in A$. Let $b = f(a)$.

Then, by definition, $f^{-1}(b) = a$. Then $f^{-1} \circ f(a) = f^{-1}(f(a)) = f^{-1}(b) = a$.

Now we will show that $f \circ f^{-1} = 1$. Let $b \in B$. Let $a = f^{-1}(b)$.

Then, by definition, $f(a) = b$. Then $f \circ f^{-1}(b) = f(f^{-1}(b)) = f(a) = b$.

Conclusion,

$$\boxed{f^{-1}\ inverse\ function\ to\ the\ function\ f}$$

Theorem 2. Let $f\,:\,A \to B$ have an inverse. Then f is bijective.

Proof. Let $f : A \to B$ have an inverse $f^{-1} : B \to A$.

First, we will show that f is surjective. Suppose $b \in B$. Let $a = f^{-1}(b)$.

Then $f(a) = f\left(f^{-1}(b)\right) = f \circ f^{-1}(b) = 1 \cdot (b) = b$. So f is surjective.

Now, we will show that f is injective. Let $a_1, a_2 \in A$ be such that $f(a_1) = f(a_2)$.

We will show $a_1 = a_2$. Let $b = f(a_1)$. Let $a = f^{-1}(b)$.

Then,

$a_2 = 1 \cdot (a_2) = f^{-1} \circ f(a_2) = f^{-1}\left(f(a_2)\right) = f^{-1}(b) = a$.

But at the same time,

$a_1 = 1 \cdot (a_1) = f^{-1} \circ f(a_1) = f^{-1}\left(f(a_1)\right) = f^{-1}\left(f(a_2)\right) = f^{-1}(b) = a$.

Therefore, $a_1 = a_2$ and we have shown that f is injective.

Conclusion,

$$We\ have\ proved\ that\ if\ a\ function\ has\ the\ inverse, then\ it\ is\ a\ bijection.$$