# Answer to Question #74135, Math / Quantitative Methods

Determine the constants $a, b, c$ in the differentiation formula

$$y''(x) = ay(x-h) + by(x) + cy(x+h)$$

so that the method of the highest possible order and the error term of the method.

**Solution.**

We choose the coefficients $a, b, c$ so that $y''(x)$ is exact for $y(x) = x^j$ for $j = 0,1,2,\dots$

From Taylor series:

$$y(x-h) = y(x) - hy'(x) + \frac{1}{2!}h^2y''(x) - \frac{1}{3!}h^3y'''(x) + \frac{1}{4!}h^4y^{(4)}(x) + \cdots$$

$$y(x+h) = y(x) + hy'(x) + \frac{1}{2!}h^2y''(x) + \frac{1}{3!}h^3y'''(x) + \frac{1}{4!}h^4y^{(4)}(x) + \cdots$$

If $y(x) = 1$ then

$a + b + c = 0$

If $y(x) = x$ then

$$h(a-c) = 0$$

If $y(x) = x^2$ then

$$\frac{h^2(a+c)}{2} = 1$$

Thus:

$$a = c = \frac{1}{h^2}\ ; b = -\frac{2}{h^2}$$

The error:

$$\varepsilon = y''(x) - D_h^{(2)}y(x)$$

$$D_h^{(2)}y(x) = \frac{y(x-h) - 2y(x) + y(x+h)}{h^2}$$

Using the Taylor series approach:

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$$\varepsilon \approx y''(x) - \frac{y(x-h) - 2y(x) + y(x+h)}{h^2} = y''(x) - y''(x) - 2 \cdot \frac{1}{4!} h^2 y^{(4)}(x)$$

$$\varepsilon \approx -\frac{2}{24} h^2 y^{(4)}(x) = -\frac{h^2}{12} y^{(4)}(x)$$