Answer on Question #74059 – Math – Differential Equations

Question

A tightly stretched string with fixed end points x=0 and x=1 is initially in a position given by $y = y_0 \sin^3\left(\frac{\pi x}{l}\right)$ it is released from rest from the initial position. Find the displacement $y(x,t)$.

Solution

The equation of the string is $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ (1)

Let y = X(x)T(t)

X is a function of x and T is the function of t.

$$\frac{\partial y}{\partial t} = \frac{\partial(XT)}{\partial t} = XT' \quad \frac{\partial^2 y}{\partial t^2} = XT''$$

Similarly we calculate $\frac{\partial^2 y}{\partial x^2}$ and we get

$$\frac{\partial y}{\partial x} = \frac{\partial(XT)}{\partial x} = X'T \quad \frac{\partial^2 y}{\partial x^2} = X''T$$

Putting into equation (1), we get

$$XT'' = c^2 \; TX''$$

$$\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} \quad = K = (-p^2) \qquad (say) \quad (2)$$

Using by separation of variable we solve these two cases separately

$$\frac{1}{c^2}\frac{T''}{T} = -p^2 \quad and \quad \frac{X''}{X} = -p^2 \qquad (3)$$

$$T'' = -c^2 p^2 T \quad and \quad , X'' = -p^2 X$$

$$D' = \pm pci \qquad and \qquad D = \pm pi$$

$$T = C_1 \cos cpt + C_2 \sin cpt \quad X = C_3 \cos px + C_4 \sin px$$

General solution of equation (3) and put into equation (1)

$$y(x,t) = (C_1 \cos cpt + C_2 \sin cpt)(C_3 \cos px + C_4 \sin px) \quad (3^*)$$

Given boundary conditions are

$$y(0,t)=0\ ,\ y(l,t)=0\ \left(\frac{\partial y}{\partial t}\right)_{t=0}=0\,,\ y(x,0)=y_0\sin^3\left(\frac{\pi x}{l}\right) \qquad (4)$$

Appling boundaries conditions in equation (3*)

$$y(0,t)=0=(C_1\cos cpt+C_2\sin cpt)(C_3)$$

So C_3=0

Equation (3*) will become $y(x,t)=(C_1\cos cpt+C_2\sin cpt)(C_4\sin px) \qquad (5)$

Again $y(l,t)=0$ $y(x,t)=(C_1\cos cpt+C_2\sin cpt)(C_4\sin pl)$

$$\sin pl=0=\sin p\pi$$

$$We\quad get\quad p=\left(\frac{\pi x}{l}\right)\quad n\in I$$

Hence from (5) $y(x,t)=(C_1\cos\frac{n\pi ct}{l}+C_2\sin\frac{n\pi ct}{l})(C_4\sin\frac{n\pi x}{l})$

$$\left(\frac{\partial y}{\partial t}\right)=\frac{n\pi c}{l}\left[(-C_1\sin\frac{n\pi ct}{l}+C_2\cos\frac{n\pi ct}{l})(C_4\sin\frac{n\pi x}{l})\right]$$

At t=0

$$\left(\frac{\partial y}{\partial t}\right)=0=\frac{n\pi c}{l}\left[C_2C_4\sin\frac{n\pi x}{l})\right]\quad hence\quad c_2=0$$

Hence

$$y(x,t)=\left[(C_1C_4\cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l})\right],\ y(x,t)=\left[(b_n\cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l})\right]\quad here\ b_n=C_1C_2$$

Most general solution is

$$y(x,t)=\left[\sum_{n=1}^{\infty}b_n\cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l})\right] \qquad (6)$$

$$y(x,0)=y_0\sin^3\left(\frac{\pi x}{l}\right)=\left[\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}\right]$$

$$y_0=\frac{1}{4}\left(3\sin\frac{\pi x}{l}-\sin\frac{3.\pi x}{l}\right)=b_1\sin\frac{1.\pi x}{l}+b_2\sin\frac{2.\pi x}{l}+b_3\sin\frac{3\pi x}{l}+\ldots\ldots\ldots$$

Comparing and we get

$$b_1 = \frac{3y_0}{4}, b_2 = 0, b_3 = \frac{-y_0}{4}, b_4 = 0, b_5 = 0.......$$

Finally equation (6) will be

$$y(x,t) = \left[\frac{3y_0}{4}\cos\frac{\pi ct}{l}\sin\frac{n\pi x}{l} - \frac{y_0}{4}\cos\frac{3\pi ct}{l}\sin\frac{3\pi x}{l}\right]$$

which is the required displacement.