Answer on Question #74014 – Math – Quantitative Methods

Question

For the linear system of equations [1 2 - 2,1 1 1, 2 2 1][x y z] =[1 3 5] set up the gauss - jacobi and gauss - seidel iteration schemes in matrix form. also check the convergence of the two schemes.

Solution

Consider a system of linear equations $\mathrm{Au} = \mathrm{b}$ with

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}, u = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, b = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

1. The matrix form of Jacobi iterative method is

$$u^{k+1} = D^{-1}(b - Ru^k)$$

where u(k) is the kth approximation or iteration of u and u(k+1) is the next or k+1 iteration of u and A=D+R.

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 & -2 \\ 1 & 0 & 1 \\ 2 & 2 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = D^{-1}, \qquad R = \begin{bmatrix} 0 & 2 & -2 \\ 1 & 0 & 1 \\ 2 & 2 & 0 \end{bmatrix}$$

The convergence condition is: $\rho(\mathrm{B}) < 1$ where $B = D^{-1}R$ and $\rho(B) = \max\{|l_1|, |l_2|, |l_3|\}$, l1,l2,l3 - eigenvalues of a matrix B.

$$l_{1,2,3} = \begin{bmatrix} 0.000005 + 0.000009i \\ 0.000005 - 0.000009i \\ -0.000011 \end{bmatrix} \to \max\{|l_1|, |l_2|, |l_3|\} = 0.000011 < 1$$

$$u^{k+1} = D^{-1}(b - Ru^k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 0 & 2 & -2 \\ 1 & 0 & 1 \\ 2 & 2 & 0 \end{bmatrix} \cdot u^k \right)$$
$$= \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 0 & 2 & -2 \\ 1 & 0 & 1 \\ 2 & 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} x^k \\ y^k \\ z^k \end{bmatrix} = \begin{bmatrix} 1 - 2y^k + 2z^k \\ 3 - x^k - z^k \\ 5 - 2x^k - 2y^k \end{bmatrix}$$

2. The matrix form of gauss – seidel iterative method is

$$u^{k+1} = L^{-1}(b - Mu^k)$$

where u(k) is the kth approximation or iteration of u and u(k+1) is the next or k+1 iteration of u and A=L+M.

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}, L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}, M = \begin{bmatrix} 0 & 2 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The convergence condition is: $\rho(\mathrm{B}) < 1$ where $B = L^{-1}M$ and $\rho(B) = \max\{|l_1|, |l_2|, |l_3|\}$, l1,l2,l3 - eigenvalues of a matrix B.

$$l_{1,2,3} = \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix} \to \max\{|l_1|, |l_2|, |l_3|\} = 2 > 1$$

$$u^{k+1} = L^{-1}(b - Mu^k) = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 0 & 2 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot u^k \right)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 - 2y^k + 2z^k \\ 3 - z^k \\ 5 \end{bmatrix} = \begin{bmatrix} 1 - 2y^k + 2z^k \\ 2 + 2y^k - 3z^k \\ 2z^k - 1 \end{bmatrix}$$

Answer:

1. jacobi scheme: $\begin{bmatrix} x^{k+1} \\ y^{k+1} \\ z^{k+1} \end{bmatrix} = \begin{bmatrix} 1 - 2y^k + 2z^k \\ 3 - x^k - z^k \\ 5 - 2x^k - 2y^k \end{bmatrix}$, the convergence condition is satisfied

2. gauss – seidel scheme: $\begin{bmatrix} x^{k+1} \\ y^{k+1} \\ z^{k+1} \end{bmatrix} = \begin{bmatrix} 1 - 2y^k + 2z^k \\ 2 + 2y^k - 3z^k \\ 2z^k - 1 \end{bmatrix}$, the convergence condition is not satisfied