

Answer on Question #73768 – Math – Analytic Geometry

QUESTION

Check whether or not the conicoid represented by

$$5x^2 + 4y^2 - 4yz + 2xz + 2x - 4y - 8z + 2 = 0$$

is central or not. If it is, transform the equation by shifting the origin to the center. Else, change any one coefficient to make the equation that of a central conicoid.

SOLUTION

THEOREM 1

The origin $O(0,0,0)$ is a centre of the conicoid

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$$

If and only if

$$u = v = w = 0$$

THEOREM 2

A conicoid S, given by equation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$$

has the point $P(x_0, y_0, z_0)$ as the center if and only if

$$\begin{cases} ax_0 + hy_0 + gz_0 + u = 0 \\ hx_0 + by_0 + fz_0 + v = 0 \\ gx_0 + fy_0 + cz_0 + w = 0 \end{cases}$$

Using these two theorems, we can solve the problem posed.

We reduce the equation from the problem to the standard form:

$$5x^2 + 4y^2 - 4yz + 2xz + 2x - 4y - 8z + 2 = 0 \rightarrow$$
$$\underbrace{5}_a x^2 + \underbrace{4}_b y^2 + \underbrace{0}_c z^2 + 2 \cdot \underbrace{(-2)}_f yz + 2 \cdot \underbrace{(1)}_g xz + \underbrace{0}_h xy +$$

$$+2 \cdot \underbrace{(1)}_u x + 2 \cdot \underbrace{(-2)}_v y + 2 \cdot \underbrace{(-4)}_w z + 2 = 0$$

As we can see,

$$\begin{cases} u = 1 \neq 0 \\ v = -2 \neq 0 \\ w = -4 \neq 0 \end{cases} \rightarrow O(0,0,0) \text{ is not the center of this surface (by THEOREM 1)}$$

In order to find the center, it is necessary to solve the system

$$\begin{cases} ax_0 + hy_0 + gz_0 + u = 0 \\ hx_0 + by_0 + fz_0 + v = 0 \\ gx_0 + fy_0 + cz_0 + w = 0 \end{cases} \rightarrow (\text{THEOREM 2})$$

In our case,

$$\begin{cases} 5x_0 + 0y_0 + 1z_0 + 1 = 0 \\ 0x_0 + 4y_0 + (-2)z_0 + (-2) = 0 \\ 1x_0 + (-2)y_0 + 0z_0 + (-4) = 0 \end{cases} \rightarrow \begin{cases} 5x_0 + 0y_0 + 1z_0 = -1 \\ 0x_0 + 4y_0 + (-2)z_0 = 2 \\ 1x_0 + (-2)y_0 + 0z_0 = 4 \end{cases}$$

This system will be solved using Cramer's rule

(More information: https://en.wikipedia.org/wiki/Cramer%27s_rule)

$$\Delta = \begin{vmatrix} 5 & 0 & 1 \\ 0 & 4 & -2 \\ 1 & -2 & 0 \end{vmatrix} =$$

$$= 5 \cdot 4 \cdot 0 + 0 \cdot (-2) \cdot 1 + 0 \cdot (-2) \cdot 1 - 1 \cdot 1 \cdot 4 - 5 \cdot (-2) \cdot (-2) - 0 \cdot 0 \cdot 0 =$$

$$= 0 + 0 + 0 - 4 - 20 - 0 = -24$$

$$\boxed{\Delta = -24}$$

$$\Delta_{x_0} = \begin{vmatrix} -1 & 0 & 1 \\ 2 & 4 & -2 \\ 4 & -2 & 0 \end{vmatrix} =$$

$$= (-1) \cdot 4 \cdot 0 + 2 \cdot (-2) \cdot 1 + 0 \cdot (-2) \cdot 4 - 1 \cdot 4 \cdot 4 - (-1) \cdot (-2) \cdot (-2) - 2 \cdot 0 \cdot 0 =$$
$$= 0 - 4 + 0 - 16 + 4 - 0 = -16$$

$$\boxed{\Delta_{x_0} = -16}$$

$$x_0 = \frac{\Delta_{x_0}}{\Delta} = \frac{-16}{-24} = \frac{2 \cdot 8}{3 \cdot 8} = \frac{2}{3} \rightarrow \boxed{x_0 = \frac{2}{3} = 0.(6) \approx 0.67}$$

$$\Delta_{y_0} = \begin{vmatrix} 5 & -1 & 1 \\ 0 & 2 & -2 \\ 1 & 4 & 0 \end{vmatrix} =$$

$$= 5 \cdot 2 \cdot 0 + (-1) \cdot (-2) \cdot 1 + 0 \cdot 1 \cdot 4 - 1 \cdot 2 \cdot 1 - 5 \cdot 4 \cdot (-2) - (-1) \cdot 0 \cdot 0 =$$
$$= 0 + 2 + 0 - 2 + 40 - 0 = 40$$

$$\boxed{\Delta_{y_0} = 40}$$

$$y_0 = \frac{\Delta_{y_0}}{\Delta} = \frac{40}{-24} = -\frac{5 \cdot 8}{3 \cdot 8} = -\frac{5}{3} \rightarrow \boxed{y_0 = -\frac{5}{3} = -1.(6) \approx -1.67}$$

$$\Delta_{z_0} = \begin{vmatrix} 5 & 0 & -1 \\ 0 & 4 & 2 \\ 1 & -2 & 4 \end{vmatrix} =$$

$$= 5 \cdot 4 \cdot 4 + (-1) \cdot (-2) \cdot 0 + 0 \cdot 1 \cdot 2 - 1 \cdot 4 \cdot (-1) - 5 \cdot 2 \cdot (-2) - 4 \cdot 0 \cdot 0 =$$
$$= 80 - 0 + 0 + 4 + 20 - 0 = 104$$

$$\boxed{\Delta_{z_0} = 104}$$

$$z_0 = \frac{\Delta_{z_0}}{\Delta} = \frac{104}{-24} = -\frac{8 \cdot 13}{8 \cdot 3} = -\frac{13}{3} \rightarrow \boxed{z_0 = -\frac{13}{3} = -4.(3) \approx -4.33}$$

Conclusion,

$$P\left(\frac{2}{3}, -\frac{5}{3}, -\frac{13}{3}\right) \text{ is the center of the conicoid}$$

In order for the conicoid to become central, it is necessary to apply a shift of the form

$$\begin{cases} X = x - \frac{2}{3} \\ Y = y + \frac{5}{3} \\ Z = z + \frac{13}{3} \end{cases} \rightarrow \begin{cases} x = X + \frac{2}{3} \\ y = Y - \frac{5}{3} \\ z = Z - \frac{13}{3} \end{cases}$$

Then,

$$\begin{aligned} & 5x^2 + 4y^2 - 4yz + 2xz + 2x - 4y - 8z + 2 = \\ & = 5\left(X + \frac{2}{3}\right)^2 + 4\left(Y - \frac{5}{3}\right)^2 - 4\left(Y - \frac{5}{3}\right)\left(Z - \frac{13}{3}\right) + 2\left(X + \frac{2}{3}\right)\left(Z - \frac{13}{3}\right) + \\ & \quad + 2\left(X + \frac{2}{3}\right) - 4\left(Y - \frac{5}{3}\right) - 8\left(Z - \frac{13}{3}\right) + 2 = \\ & = 5\left(X^2 + 2X \cdot \frac{2}{3} + \frac{4}{9}\right) + 4\left(Y^2 - 2Y \cdot \frac{5}{3} + \frac{25}{9}\right) - 4\left(YZ - \frac{13}{3}Y - \frac{5}{3}Z + \frac{65}{9}\right) + \\ & \quad + 2\left(XZ - \frac{13}{3}X + \frac{2}{3}Z - \frac{26}{9}\right) + 2X + \frac{4}{3} - 4Y + \frac{20}{3} - 8Z + \frac{104}{3} + 2 = \\ & = 5X^2 + \frac{20}{3}X + \frac{20}{9} + 4Y^2 - \frac{40}{3}Y + \frac{100}{9} - 4YZ + \frac{52}{3}Y + \frac{20}{3}Z - \frac{260}{9} + \\ & \quad + 2XZ - \frac{26}{3}X + \frac{4}{3}Z - \frac{52}{9} + 2X + \frac{4}{3} - 4Y + \frac{20}{3} - 8Z + \frac{104}{3} + 2 = \\ & = 5X^2 + 4Y^2 - 4YZ + 2XZ + X\left(\frac{20}{3} - \frac{26}{3} + 2\right) + Y\left(-\frac{40}{3} + \frac{52}{3} - 4\right) + \\ & \quad + Z\left(\frac{20}{3} + \frac{4}{3} - 8\right) + \left(\frac{20}{9} + \frac{100}{9} - \frac{260}{9} - \frac{52}{9} + \frac{4}{3} + \frac{20}{3} + \frac{104}{3} + 2\right) = \end{aligned}$$

$$\begin{aligned}
&= 5X^2 + 4Y^2 - 4YZ + 2XZ + X\left(\frac{20 - 26 + 6}{3}\right) + Y\left(\frac{-40 + 52 - 12}{3}\right) + \\
&\quad + Z\left(\frac{20 + 4 - 24}{3}\right) + \left(\frac{20 + 100 - 260 - 52}{9} + \frac{4 + 20 + 104}{3} + 2\right) = \\
&= 5X^2 + 4Y^2 - 4YZ + 2XZ + 0 \cdot X + 0 \cdot Y + 0 \cdot Z + \left(-\frac{192}{9} + \frac{128}{3} + 2\right) = \\
&= 5X^2 + 4Y^2 - 4YZ + 2XZ + 0 \cdot X + 0 \cdot Y + 0 \cdot Z + \left(\frac{-192 + 3 \cdot 128 + 2 \cdot 9}{9}\right) = \\
&= 5X^2 + 4Y^2 - 4YZ + 2XZ + 0 \cdot X + 0 \cdot Y + 0 \cdot Z + \left(\frac{-192 + 384 + 18}{9}\right) = \\
&= 5X^2 + 4Y^2 - 4YZ + 2XZ + 0 \cdot X + 0 \cdot Y + 0 \cdot Z + \frac{210}{9} = \\
&= 5X^2 + 4Y^2 - 4YZ + 2XZ + 0 \cdot X + 0 \cdot Y + 0 \cdot Z + \frac{70}{3}
\end{aligned}$$

Conclusion,

$$5x^2 + 4y^2 - 4yz + 2xz + 2x - 4y - 8z + 2 = 0 \rightarrow 5X^2 + 4Y^2 - 4YZ + 2XZ + \frac{70}{3} = 0$$

ANSWER

1) The conicoid represented by

$$5x^2 + 4y^2 - 4yz + 2xz + 2x - 4y - 8z + 2 = 0$$

Is'nt central.

2) The point P is centre of the conicoid.

$$p.P \left(\frac{2}{3}, -\frac{5}{3}, -\frac{13}{3}\right) - \text{Center of the conicoid}$$

3) it is necessary to apply a shift of the form

$$\begin{cases} x = X + \frac{2}{3} \\ y = Y - \frac{5}{3} \\ z = Z - \frac{13}{3} \end{cases}$$

to make the conicoid central.

$$5x^2 + 4y^2 - 4yz + 2xz + 2x - 4y - 8z + 2 = 0 \rightarrow 5X^2 + 4Y^2 - 4YZ + 2XZ + \frac{70}{3} = 0$$