Answer on Question #73767 – Math – Analytic Geometry

QUESTION

Check whether or not the conicoid represented by

$$5x^2+4y^2-4yz+2xz+2x-4y-8z+2=0$$

is central or not. If it is, transform the equation by shifting the origin to the center. Else, change any one coefficient to make the equation that of a central conicoid.

SOLUTION

<u>THEOREM 1</u>

The origin $p.O(0,0,0)$ is a centre of the conicoid

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$$

If and only if

$$u=v=w=0$$

<u>THEOREM 2</u>

A conicoid S, given by equation

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$$

has the point $P(x_0,y_0,z_0)$ as the center if and only if

$$\begin{cases} ax_0+hy_0+gz_0+u=0 \\ hx_0+by_0+fz_0+v=0 \\ gx_0+fy_0+cz_0+w=0 \end{cases}$$

Using these two theorems, we can solve the problem posed.

We reduce the equation from the problem to the standard form:

$$5x^2+4y^2-4yz+2xz+2x-4y-8z+2=0\rightarrow$$

$$\underbrace{5}_{a}x^2+\underbrace{4}_{b}y^2+\underbrace{0}_{c}z^2+2\cdot\underbrace{(-2)}_{f}yz+2\cdot\underbrace{(1)}_{g}xz+\underbrace{0}_{h}xy+$$

$$+2\cdot\underbrace{(1)}_{u}x+2\cdot\underbrace{(-2)}_{v}y+2\cdot\underbrace{(-4)}_{w}z+2=0$$

As we can see,

$$\begin{cases} u=1\neq 0\\ v=-2\neq 0\\ w=-4\neq 0\end{cases}\rightarrow O(0,0,0)\ is\ not\ the\ center\ of\ this\ surface(\ by\ THEOREM\ 1)$$

In order to find the center, it is necessary to solve the system

$$\begin{cases} ax_0+hy_0+gz_0+u=0\\ hx_0+by_0+fz_0+v=0\\ gx_0+fy_0+cz_0+w=0\end{cases}\rightarrow (THEOREM\ 2)$$

In our case,

$$\begin{cases} 5x_0+0y_0+1z_0+1=0\\ 0x_0+4y_0+(-2)z_0+(-2)=0\\ 1x_0+(-2)y_0+0z_0+(-4)=0\end{cases}\rightarrow\begin{cases} 5x_0+0y_0+1z_0=-1\\ 0x_0+4y_0+(-2)z_0=2\\ 1x_0+(-2)y_0+0z_0=4\end{cases}$$

This system will be solved using Cramer's rule

(More information: https://en.wikipedia.org/wiki/Cramer%27s_rule)

$$\Delta=\begin{vmatrix}5 & 0 & 1\\ 0 & 4 & -2\\ 1 & -2 & 0\end{vmatrix}=$$

$$=5\cdot 4\cdot 0+0\cdot(-2)\cdot 1+0\cdot(-2)\cdot 1-1\cdot 1\cdot 4-5\cdot(-2)\cdot(-2)-0\cdot 0\cdot 0=$$

$$=0+0+0-4-20-0=-24$$

$$\boxed{\Delta=-24}$$

$$\Delta_{x_0} = \begin{vmatrix} -1 & 0 & 1 \\ 2 & 4 & -2 \\ 4 & -2 & 0 \end{vmatrix} =$$

$$= (-1) \cdot 4 \cdot 0 + 2 \cdot (-2) \cdot 1 + 0 \cdot (-2) \cdot 4 - 1 \cdot 4 \cdot 4 - (-1) \cdot (-2) \cdot (-2) - 2 \cdot 0 \cdot 0 =$$

$$= 0 - 4 + 0 - 16 + 4 - 0 = -16$$

$$\boxed{\Delta_{\mathrm{x}_0} = -16}$$

$$x_0 = \frac{\Delta_{x_0}}{\Delta} = \frac{-16}{-24} = \frac{2 \cdot 8}{3 \cdot 8} = \frac{2}{3} \rightarrow \boxed{x_0 = \frac{2}{3} = 0.(6) \approx 0.67}$$

$$\Delta_{y_0} = \begin{vmatrix} 5 & -1 & 1 \\ 0 & 2 & -2 \\ 1 & 4 & 0 \end{vmatrix} =$$

$$= 5 \cdot 2 \cdot 0 + (-1) \cdot (-2) \cdot 1 + 0 \cdot 1 \cdot 4 - 1 \cdot 2 \cdot 1 - 5 \cdot 4 \cdot (-2) - (-1) \cdot 0 \cdot 0 =$$

$$= 0 + 2 + 0 - 2 + 40 - 0 = 40$$

$$\boxed{\Delta_{\mathrm{y}_0} = 40}$$

$$y_0 = \frac{\Delta_{y_0}}{\Delta} = \frac{40}{-24} = -\frac{5 \cdot 8}{3 \cdot 8} = -\frac{5}{3} \rightarrow \boxed{y_0 = -\frac{5}{3} = -1.(6) \approx -1.67}$$

$$\Delta_{z_0} = \begin{vmatrix} 5 & 0 & -1 \\ 0 & 4 & 2 \\ 1 & -2 & 4 \end{vmatrix} =$$

$$= 5 \cdot 4 \cdot 4 + (-1) \cdot (-2) \cdot 0 + 0 \cdot 1 \cdot 2 - 1 \cdot 4 \cdot (-1) - 5 \cdot 2 \cdot (-2) - 4 \cdot 0 \cdot 0 =$$

$$= 80 - 0 + 0 + 4 + 20 - 0 = 104$$

$$\boxed{\Delta_{\mathrm{z}_0} = 104}$$

$$z_0 = \frac{\Delta_{z_0}}{\Delta} = \frac{104}{-24} = -\frac{8 \cdot 13}{8 \cdot 3} = -\frac{13}{3} \rightarrow \boxed{z_0 = -\frac{13}{3} = -4.(3) \approx -4.33}$$

Conclusion,

$$\boxed{P\left(\frac{2}{3},-\frac{5}{3},-\frac{13}{3}\right)\ is\ the\ center\ of\ the\ conicoid}$$

In order for the conicoid to become central, it is necessary to apply a shift of the form

$$\begin{cases} X = x - \frac{2}{3} \\ Y = y + \frac{5}{3} \\ Z = z + \frac{13}{3} \end{cases} \rightarrow \begin{cases} x = X + \frac{2}{3} \\ y = Y - \frac{5}{3} \\ z = Z - \frac{13}{3} \end{cases}$$

Then,

$$5x^2 + 4y^2 - 4yz + 2xz + 2x - 4y - 8z + 2 =$$

$$= 5\left(X + \frac{2}{3}\right)^2 + 4\left(Y - \frac{5}{3}\right)^2 - 4\left(Y - \frac{5}{3}\right)\left(Z - \frac{13}{3}\right) + 2\left(X + \frac{2}{3}\right)\left(Z - \frac{13}{3}\right) +$$

$$+2\left(X + \frac{2}{3}\right) - 4\left(Y - \frac{5}{3}\right) - 8\left(Z - \frac{13}{3}\right) + 2 =$$

$$= 5\left(X^2 + 2X \cdot \frac{2}{3} + \frac{4}{9}\right) + 4\left(Y^2 - 2Y \cdot \frac{5}{3} + \frac{25}{9}\right) - 4\left(YZ - \frac{13}{3}Y - \frac{5}{3}Z + \frac{65}{9}\right) +$$

$$+2\left(XZ - \frac{13}{3}X + \frac{2}{3}Z - \frac{26}{9}\right) + 2X + \frac{4}{3} - 4Y + \frac{20}{3} - 8Z + \frac{104}{3} + 2 =$$

$$= 5X^2 + \frac{20}{3}X + \frac{20}{9} + 4Y^2 - \frac{40}{3}Y + \frac{100}{9} - 4YZ + \frac{52}{3}Y + \frac{20}{3}Z - \frac{260}{9} +$$

$$+2XZ - \frac{26}{3}X + \frac{4}{3}Z - \frac{52}{9} + 2X + \frac{4}{3} - 4Y + \frac{20}{3} - 8Z + \frac{104}{3} + 2 =$$

$$= 5X^2 + 4Y^2 - 4YZ + 2XZ + X\left(\frac{20}{3} - \frac{26}{3} + 2\right) + Y\left(-\frac{40}{3} + \frac{52}{3} - 4\right) +$$

$$+Z\left(\frac{20}{3} + \frac{4}{3} - 8\right) + \left(\frac{20}{9} + \frac{100}{9} - \frac{260}{9} - \frac{52}{9} + \frac{4}{3} + \frac{20}{3} + \frac{104}{3} + 2\right) =$$

$$= 5X^2 + 4Y^2 - 4YZ + 2XZ + X\left(\frac{20 - 26 + 6}{3}\right) + Y\left(\frac{-40 + 52 - 12}{3}\right) +$$

$$+Z\left(\frac{20 + 4 - 24}{3}\right) + \left(\frac{20 + 100 - 260 - 52}{9} + \frac{4 + 20 + 104}{3} + 2\right) =$$

$$= 5X^2 + 4Y^2 - 4YZ + 2XZ + 0 \cdot X + 0 \cdot Y + 0 \cdot Z + \left(-\frac{192}{9} + \frac{128}{3} + 2\right) =$$

$$= 5X^2 + 4Y^2 - 4YZ + 2XZ + 0 \cdot X + 0 \cdot Y + 0 \cdot Z + \left(\frac{-192 + 3 \cdot 128 + 2 \cdot 9}{9}\right) =$$

$$= 5X^2 + 4Y^2 - 4YZ + 2XZ + 0 \cdot X + 0 \cdot Y + 0 \cdot Z + \left(\frac{-192 + 384 + 18}{9}\right) =$$

$$= 5X^2 + 4Y^2 - 4YZ + 2XZ + 0 \cdot X + 0 \cdot Y + 0 \cdot Z + \frac{210}{9} =$$

$$= 5X^2 + 4Y^2 - 4YZ + 2XZ + 0 \cdot X + 0 \cdot Y + 0 \cdot Z + \frac{70}{3}$$

Conclusion,

$$5x^2 + 4y^2 - 4yz + 2xz + 2x - 4y - 8z + 2 = 0 \to 5X^2 + 4Y^2 - 4YZ + 2XZ + \frac{70}{3} = 0$$

ANSWER

1) The conicoid represented by

$$5x^2 + 4y^2 - 4yz + 2xz + 2x - 4y - 8z + 2 = 0$$

isn’t central.

2) The point P is centre of the conicoid.

$$P\left(\frac{2}{3}, -\frac{5}{3}, -\frac{13}{3}\right) \textit{is the center of the conicoid}$$

3) it is necessary to apply a shift of the form

$$\begin{cases} x = X + \frac{2}{3} \\ y = Y - \frac{5}{3} \\ z = Z - \frac{13}{3} \end{cases}$$

to make the conicoid central.

$$5x^2 + 4y^2 - 4yz + 2xz + 2x - 4y - 8z + 2 = 0 \rightarrow 5X^2 + 4Y^2 - 4YZ + 2XZ + \frac{70}{3} = 0$$