Problem 1. Using Green's theorem evaluate the integral

$$\int_C y^2\,dx + 3xy\,dy, \tag{1}$$

$$C = \{(x, y) \mid x^2 + y^2 = 1, y \geq 0\} \tag{2}$$

Solution Let C be a positively oriented, piecewise smooth, simple closed curve in a plane, and let D be the region bounded by C. Let $P(x, y), Q(x, y)$ be functions defined on an open region containing D and having continuous partial derivatives there, then

$$\int_C P(x, y)\,dx + Q(x, y)\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Let's evaluate (1). $P(x, y) = y^2$, $Q(x, y) = 3xy$.

$$\int_C y^2\,dx + 3xy\,dy = \iint_D (3y - 2y)\,dx\,dy = \iint_D y\,dx\,dy$$

Let's switch to polar coordinate system

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$
$$D = \{(r\cos\varphi, r\sin\varphi) \mid r \leq 1, \varphi \in [0, \pi]\}$$

Then our integral will turn into (the square of r appears because of Jacobian $J = r$)

$$\int\limits_0^\pi \sin\varphi\,d\varphi \int\limits_0^1 r^2\,dr = \cos\varphi|_\pi^0 \frac{r^3}{3}|_0^1 = 2 \cdot \frac{1}{3} = \frac{2}{3}$$

□