Answer on Question #73400 – Math – Differential Equations

Question

Show that the function $u(x,y) = tan^{-1}\left(\frac{y}{x}\right)$ is a solution of the two dimensional Laplace's equation[1].

Solution

Laplace equation is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

First let note that our function has a form $u(x,y) = f\left(\frac{y}{x}\right)$. Let rewrite Laplace equation for our particular case:

a) $\frac{\partial u}{\partial x} = f'\left(\frac{y}{x}\right) \cdot \left(-\frac{y}{x^2}\right)$
b) $\frac{\partial^2 u}{\partial x^2} = f''\left(\frac{y}{x}\right) \cdot \left(-\frac{y}{x^2}\right)^2 + f'\left(\frac{y}{x}\right) \cdot \left(\frac{2y}{x^3}\right)$
c) $\frac{\partial u}{\partial y} = f'\left(\frac{y}{x}\right) \cdot \left(\frac{1}{x}\right)$
d) $\frac{\partial^2 u}{\partial y^2} = f''\left(\frac{y}{x}\right) \cdot \left(\frac{1}{x}\right)^2$

The Laplace equation becomes:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''\left(\frac{y}{x}\right) \cdot \left(-\frac{y}{x^2}\right)^2 + f'\left(\frac{y}{x}\right) \cdot \left(\frac{2y}{x^3}\right) + f''\left(\frac{y}{x}\right) \cdot \left(\frac{1}{x}\right)^2 =$$

$$= \frac{1}{x^4}\left(f''\left(\frac{y}{x}\right) \cdot (y^2 + x^2) + 2f'\left(\frac{y}{x}\right) \cdot yx\right)$$

Now we back to our function $f(z) = tan^{-1}(z)$, and find first and second derivatives with respect to a single variable z.

a) $f'(z) = \frac{d}{dz} f(z) = \frac{d}{dz}\left(tan^{-1}(z)\right) = \frac{1}{z^2+1}$
b) $f''(z) = \frac{d^2}{dz^2} f(z) = \frac{d}{dz}\left(\frac{1}{z^2+1}\right) = -\frac{2z}{(z^2+1)^2}$

Finally, substitute $f''\left(\frac{y}{x}\right)$ and $f'\left(\frac{y}{x}\right)$ to corresponding Laplace equation and after some transformation and simplification we get:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{x^4}\left(-\frac{2 \cdot \left(\frac{y}{x}\right)}{\left(\left(\frac{y}{x}\right)^2 + 1\right)^2} \cdot (y^2 + x^2) + 2 \cdot \frac{1}{\left(\frac{y}{x}\right)^2 + 1} \cdot yx\right) =$$

$$= \frac{2}{x^4 \cdot \left(\left(\frac{y}{x}\right)^2 + 1\right)^2}\left(-\left(\frac{y}{x}\right) \cdot (y^2 + x^2) + yx \cdot \left(\left(\frac{y}{x}\right)^2 + 1\right)\right) =$$

[1] All formulas can be verified on http://www.wolframalpha.com/input/

$$= \frac{2}{x^4 \cdot \left(\left(\frac{y}{x} \right)^2 + 1 \right)^2} \left(-\frac{y^3}{x} - \frac{yx^2}{x} + \frac{y^3 x}{x^2} + yx \right) = 0 \text{ , Q.E.D}$$

One has to be careful to specify an appropriate domain.

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