**Answer on Question #72690 – Math – Differential Geometry | Topology**

**Question**

Find equation of the osculating plane and osculating circle of the curve at the given point.

$$\gamma(t) = (2\sin 3t\,, t, 2\cos 3t), (0, \pi, -2)$$

**Solution**

$\vec{r}(t) = 2\sin 3t\,\vec{i} + t\vec{j} + 2\cos 3t\,\vec{k}$

$\vec{r'}(t) = 6\cos 3t\,\vec{i} + \vec{j} - 6\sin 3t\,\vec{k}$

$|\vec{r'}(t)| = \sqrt{(6\cos 3t)^2 + (1)^2 + (-6\sin 3t)^2} =$

$= \sqrt{36\cos^2 3t + 36\sin^2 3t + 1} = \sqrt{36+1} = \sqrt{37}$

Unit tangent vector

$$\vec{T}(t) = \frac{\vec{r'}(t)}{|\vec{r'}(t)|} = \frac{6}{\sqrt{37}}\cos 3t\,\vec{i} + \frac{1}{\sqrt{37}}\vec{j} - \frac{6}{\sqrt{37}}\sin 3t\,\vec{k}$$

$$\vec{T'}(t) = -\frac{18}{\sqrt{37}}\sin 3t\,\vec{i} + (0)\vec{j} - \frac{18}{\sqrt{37}}\cos 3t\,\vec{k}$$

$$|\vec{T'}(t)| = \sqrt{\left(-\frac{18}{\sqrt{37}}\sin 3t\right)^2 + \left(-\frac{18}{\sqrt{37}}\cos 3t\right)^2} =$$

$$= \sqrt{\frac{324}{37}\sin^2 3t + \frac{324}{37}\cos^2 3t} = \frac{18}{\sqrt{37}}$$

Unit normal vector

$$\vec{N}(t) = \frac{\vec{T'}(t)}{|\vec{T'}(t)|} = -\sin 3t\,\vec{i} - \cos 3t\,\vec{k}$$

“Binormal vector”: unit normal vector to osculating plane

$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$

$$\vec{B}(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{6}{\sqrt{37}}\cos 3t & \frac{1}{\sqrt{37}} & -\frac{6}{\sqrt{37}}\sin 3t \\ -\sin 3t & 0 & -\cos 3t \end{vmatrix} =$$

$$= \vec{i}\left(-\frac{\cos 3t}{\sqrt{37}} + 0\right) + \vec{j}\left(\frac{6}{\sqrt{37}}\sin^2 3t + \frac{6}{\sqrt{37}}\cos^2 3t\right) + \vec{k}\left(0 + \frac{\sin 3t}{\sqrt{37}}\right) =$$

$$= -\frac{\cos 3t}{\sqrt{37}}\vec{i} + \frac{6}{\sqrt{37}}\vec{j} + \frac{\sin 3t}{\sqrt{37}}\vec{k}$$

Point $P(0, \pi, -2)$. Hence $t = \pi$.

$r(\pi) = \langle 0, \pi, -2\rangle$

$\vec{r'}(\pi) = 6\cos(3\pi)\,\vec{i} + \vec{j} - 6\sin(3\pi)\,\vec{k}$

$\overrightarrow{r'}(\pi) = \langle -6, 1, 0 \rangle$

$\vec{B}(t) = -\frac{\cos(3\pi)}{\sqrt{37}}\vec{\imath} + \frac{6}{\sqrt{37}}\vec{\jmath} + \frac{\sin(3\pi)}{\sqrt{37}}\vec{k}$

$\vec{B}(t) = \langle \frac{1}{\sqrt{37}}, \frac{6}{\sqrt{37}}, 0 \rangle$

Osculating plane

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

$\frac{1}{\sqrt{37}}(x - 0) + \frac{6}{\sqrt{37}}(y - \pi) + (0)(z - (-2)) = 0$

$x + 6y - 6\pi = 0$ or $x + 6y = 6\pi$

Osculating normal circle to the curve at the given point $P$ is the circle
passing through $P$
with the same unit tangent vector as curve,
the same unit normal vector as curve,
and the same curvature as curve

Properties:

Circle has radius $R$ where

$$R = \frac{1}{curvature} = \frac{1}{k(P)}$$

Center is distance $1/k(P)$ from $P$ in direction of $\vec{N}(P)$

$$\overrightarrow{r_C}(t) = \vec{r}(P) + \frac{1}{k(P)}\vec{N}(P)$$

$k(t) = \frac{\left|\overrightarrow{T'}(t)\right|}{\left|\overrightarrow{r'}(t)\right|}$

$k(t) = \frac{\frac{18}{\sqrt{37}}}{\sqrt{37}} = \frac{18}{37}$

$k(\pi) = \frac{18}{37}$

$\vec{N}(\pi) = \langle 0, 0, 1 \rangle$

$\vec{T}(\pi) = \langle -\frac{6}{\sqrt{37}}, \frac{1}{\sqrt{37}}, 0 \rangle$

Osculating circle has radius

$R = \frac{1}{18/37} = \frac{37}{18}$

Osculating circle has center

$\langle 0, \pi, -2 \rangle + \frac{37}{18}\langle 0, 0, 1 \rangle = \langle 0, \pi, \frac{1}{18} \rangle$

Osculating circle parametrized by

$$q(\theta) = \langle 0, \pi, \frac{1}{18} \rangle + \frac{37}{18}\left(\cos\theta \langle 0, 0, 1 \rangle + \sin\theta \langle -\frac{6}{\sqrt{37}}, \frac{1}{\sqrt{37}}, 0 \rangle\right)$$