Answer on Question 72688, Math / Geometry

Question

Find the vectors t, n, b at the given point to the curve.

γ(t)=(t^2,2/3 t^3,t), (1, 2/3, 1).

Solution.

We are given the curve

$$\vec{r}(t) = (t^2)\vec{\imath} + \left(\frac{2}{3}t^3\right)\vec{\jmath} + (t)\vec{k}$$

and the point (1, 2/3, 1)

Note that $t = 1$ because $x_0 = (1^2) = 1$, $y_0 = \left(\frac{2}{3} \cdot 1^3\right) = \frac{2}{3}$, $z_0 = (1) = 1$ is the coordinates of the given point.

1) Find the unit tangent vector $\vec{\tau}(t) = \frac{\vec{r'}(t)}{|\vec{r'}(t)|}$

$$\vec{r'}(t) = (2t)\vec{\imath} + (2t^2)\vec{\jmath} + (1)\vec{k}$$

$$\left|\vec{r'}(t)\right| = \sqrt{4t^2 + 4t^4 + 1} = \sqrt{(2t^2+1)^2} = 2t^2 + 1$$

Thus we get

$$\vec{\tau}(t) = \frac{\vec{r'}(t)}{\left|\vec{r'}(t)\right|} = \left(\frac{2t}{2t^2+1}\right)\vec{\imath} + \left(\frac{2t^2}{2t^2+1}\right)\vec{\jmath} + \left(\frac{1}{2t^2+1}\right)\vec{k}$$

2) Find the unit normal vector $\vec{n}(t) = \frac{\vec{\tau'}(t)}{|\vec{\tau'}(t)|}$

$$\vec{\tau'}(t) = \left(\frac{2 \cdot (2t^2+1) - 2t \cdot 4t}{(2t^2+1)^2}\right)\vec{\imath} + \left(\frac{4t \cdot (2t^2+1) - 2t^2 \cdot 4t}{(2t^2+1)^2}\right)\vec{\jmath} + \left(\frac{-4t}{(2t^2+1)^2}\right)\vec{k}$$

$$= \left(\frac{2-4t^2}{(2t^2+1)^2}\right)\vec{\imath} + \left(\frac{4t}{(2t^2+1)^2}\right)\vec{\jmath} + \left(\frac{-4t}{(2t^2+1)^2}\right)\vec{k}$$

$$\left|\vec{\tau'}(t)\right| = \frac{\sqrt{(2-4t^2)^2 + (4t)^2 + (-4t)^2}}{(2t^2+1)^2} = \frac{\sqrt{4 - 16t^2 + 16t^4 + 16t^2 + 16t^2}}{(2t^2+1)^2}$$

$$= \frac{\sqrt{4 + 16t^2 + 16t^4}}{(2t^2+1)^2} = \frac{\sqrt{4(1 + 4t^2 + 4t^4)}}{(2t^2+1)^2} = \frac{2\sqrt{(1+2t^2)^2}}{(2t^2+1)^2} = \frac{2(1+2t^2)}{(2t^2+1)^2} = \frac{2}{(2t^2+1)}$$

Thus we get

$$\vec{n}(t) = \frac{\vec{\tau'}(t)}{\left|\vec{\tau'}(t)\right|} = \frac{\left(\frac{2-4t^2}{(2t^2+1)^2}\right)\vec{\imath} + \left(\frac{4t}{(2t^2+1)^2}\right)\vec{\jmath} - \left(\frac{4t}{(2t^2+1)^2}\right)\vec{k}}{\frac{2}{(2t^2+1)}}$$

$$= \left(\frac{1-2t^2}{2t^2+1}\right)\vec{\imath} + \left(\frac{2t}{2t^2+1}\right)\vec{\jmath} - \left(\frac{2t}{2t^2+1}\right)\vec{k}$$

3) Find the unit binormal vector $\vec{b}(t) = \vec{\tau} \times \vec{n}$

$$\vec{b}(t) = \vec{\tau} \times \vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{2t}{2t^2+1} & \frac{2t^2}{2t^2+1} & \frac{1}{2t^2+1} \\ \frac{1-2t^2}{2t^2+1} & \frac{2t}{2t^2+1} & -\frac{2t}{2t^2+1} \end{vmatrix}$$

$$= \vec{i}\left(-\frac{4t^3}{(2t^2+1)^2} - \frac{2t}{(2t^2+1)^2}\right) - \vec{j}\left(-\frac{4t^2}{(2t^2+1)^2} - \frac{1-2t^2}{(2t^2+1)^2}\right) + \vec{k}\left(\frac{4t^2}{(2t^2+1)^2} - \frac{2t^2-4t^4}{(2t^2+1)^2}\right)$$

$$= -\left(\frac{2t(2t^2+1)}{(2t^2+1)^2}\right)\vec{i} + \left(\frac{2t^2+1}{(2t^2+1)^2}\right)\vec{j} + \vec{k}\left(\frac{2t^2(2t^2+1)}{(2t^2+1)^2}\right)$$

$$= -\left(\frac{2t}{2t^2+1}\right)\vec{i} + \left(\frac{1}{2t^2+1}\right)\vec{j} + \left(\frac{2t^2}{2t^2+1}\right)\vec{k}$$

Thus, for $t = 1$ that is at the given point (1, 2/3, 1) we have:

$$\vec{\tau}(1,2/3,1) = \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{1}{3}\vec{k}$$

$$\vec{n}(1,2/3,1) = -\frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} - \frac{2}{3}\vec{k}$$

$$\vec{b}(1,2/3,1) = -\frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} + \frac{2}{3}\vec{k}$$

Answer: at the given point:

the unit tangent vector is

$$\vec{\tau}(1,2/3,1) = \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{1}{3}\vec{k}$$

the unit normal vector is

$$\vec{n}(1,2/3,1) = -\frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} - \frac{2}{3}\vec{k}$$

the unit binormal vector is

$$\vec{b}(1,2/3,1) = -\frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} + \frac{2}{3}\vec{k}$$