Hahn-Banach Theorem :

Statement: Let X be a real vector space and p a sub linear functional on X. Furthermore, let f be a linear functional which is defined on a subspace Z of X and satisfies

$$f(x) \le g(x) \quad \forall \quad x \in z$$

Then f has a linear extension $\tilde{f}(x)$ from Z to X satisfying

$\tilde{f}(x) \le f(x)$ , $\forall\ x \in X$, (1)

that is, $\tilde{f}(x)$ is a linear functional on X, satisfies (1) on X and

$\tilde{f}(x) \le f(x) \quad \forall \quad x \in z$ .

Proof: Now we discuss stepwise, we shall prove:

(I) The set E of all linear extensions g of f satisfying $g(x) \le p(x)$ on their domain D can be partially ordered and Zorn's lemma yields a maximal element $\tilde{f}(x)$ of E.

(II) . $\tilde{f}(x)$ is defined on the entire space X.

(III) An auxiliary relation which was used in (b). We start with part

Now we start part (i)

Let E be the set of all linear extensions g of f which satisfy the condition $g(x) \le p(x) \ \forall \ x \in D,$

Clearly, $E \ne \phi$ since $f \in E$ . On E we can define a partial ordering by $g(x) \le h(x)$ meaning h is an extension of g, that is, by definition, $D(h) \supset D(g)$ and h(x) = g(x) for every $x \in D(g)$. For any chain $C \subset E$ we now define $\hat{g}(x)$ by $\hat{g}(x) = g(x)$ g(x) if $x \in D(g)\ g \in C$. $\hat{g}(x)$ is a linear functional, the domain being

$$D(\hat{g}) = \bigcup_{g \subset C} D(g),$$

which is a vector space since C is a chain. The definition of $\hat{g}(x)$ is unambiguous. Indeed, for an $x \in D(g_1) \cap D(g_2)$ with $g_1, g_2 \in C$

we have $g_1(x) = g_2(x)$ since C is a chain,

so that

$g_1(x) \le g_2(x)$ or $g_2(x) \le g_1(x)$

$g \le \acute{g}$ for all g E C.

Hence g is an upper bound of C. Since $C \subset E$ was arbitrary, Zorn's lemma thus implies that E has a maximal element $\acute{f}$ By the definition of E, this is a linear extension of f which satisfies

$$\acute{f}(x) \le p(x) \quad \forall\ x \in D(\acute{f}), \qquad (2)$$

Now we start part (ii)

We now show that $D(\acute{f}),$ is all of X. Suppose that this is false. Then we can choose a

$y_1 \in X - D(f)$ and consider the subspace $y_1$ of X spanned by $D(\acute{f})$ and $y_1$. Note that $y_1 \neq 0$ since $0 \in D(\acute{f})$ $x \in Y_1$

It can be written $x = y + \alpha y_1$

This representation is unique.

In fact, $y + \alpha y_1 = \acute{y} + \beta y_1$ with $\acute{f} \in D(\acute{f})$ implies $y - \acute{y} = \alpha y_1 - \beta y_1$

$y - \acute{y} \in D(f)$ when $y_1 \notin D(\acute{f})$

, so that the only solution is $y - \acute{y} = 0$ and $\alpha - \beta = 0$

This means uniqueness. A functional $g_1$ on $y_1$ is defined by

$$g_1(y + \alpha y_1) = \acute{f}(y) + \alpha c \qquad (3)$$

where c is any real constant. It is not difficult to see that $g_1$ is linear. Furthermore, for $\alpha = 0$

we have $g_1(y) = \acute{f}(y)$. Hence $g_1$ is a proper extension of $(\acute{f})$. that is, an extension such that $D(\acute{f})$ is a proper subset of $D(g_1)$ Consequently, if we can prove that $g_1 \in E$ by showing that

$g_1(x) \le p(x) \;\; \forall \; x \in D(g_1),$

this will contradict the maximalist of $(\tilde{f})$, so that $D(\tilde{f}) \ne X$ and $D(\tilde{f}) = X$ is true.

Now we start part (iii)

Accordingly, we must finally show that $g_1$with a suitable c in above equations. We consider any Y and z in $D(\tilde{f})$. From (2) and (1) we obtain

$$\tilde{f}(y) - \tilde{f}(z) = \tilde{f}(y-z) \le p(y-z)$$
$$= p(y + y_1 - y_1 - z)$$
$$\le p(y + y_1) + p(-y_1 - z)$$

Taking the last term to the left and the term $(\tilde{f})$ to the right,

we have

$$-p(-y_1 - z) - \tilde{f}(z) \le p(y + y_1) - \tilde{f}(y)$$

where $Y_I$ is fixed. Since Y does not appear on the left and z not on the right, the inequality continues to hold if we take the supremum over $z \in D(\tilde{f})$ on the left and the infimum over $y \in D(\tilde{f})$ on the right, call it $m_I$

$m_0 \le m_1$ and $m_0 \le c \le m_1$

we have from (7) (8a) (8b) –

$$-p(-y_1 - z) - \tilde{f}(z) \le c$$

$$c \le p(y_1 + y_1) - \tilde{f}(y) \qquad z \in D(\tilde{f}) \;\; y \in D(\tilde{f})$$

We prove (6) first for negative $\alpha$ in equation and then for positive $\alpha$.

$\alpha < 0$ and put z=$\alpha^{-1} y$

$$-p(-y_1 - \alpha^{-1} y) - \tilde{f}(\alpha^{-1} y) \le c$$

Multiplication by -$\alpha$ > 0 gives

$$\alpha p(-y_1 - \alpha^{-1} y) + \tilde{f}(y) \le -\alpha c$$

From this and (5), using $y - \alpha y_1 = x$, we obtain the desired inequality

$$g_1(x) = \tilde{f}(y) + \alpha c \le -\alpha p(-y_1 - \alpha^{-1} y) = p(\alpha y_1 + y) = p(x)$$

For $\alpha = 0$ we have $x \in D(\tilde{f})$ and nothing to prove. For $\alpha > 0$ we use (8b) with y replaced by $\alpha^{-1} y$ to get

$$c \le p(y_1 + \alpha^{-1} y) - \tilde{f}(\alpha^{-1} y)$$

Multiplication by $\alpha > 0$ gives

$$\alpha c \le -\alpha p(y_1 + \alpha^{-1} y) - \tilde{f}(y) = p(x) - \tilde{f}(y).$$

From this and (5),

$$g_1(x) = \tilde{f}(y) + \alpha c \le p(x)$$

Hence proved

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