Zorn's lemma

Statement: Let M≠ 0 be a partially ordered set. Suppose that every chain $C \subset P$ has an upper bound. Then P has at least one maximal element.

Proof: For each $A \in P,$ Let TA be the set defined by

$$T_A = \begin{cases} \{A\} & if \;\; A \;\; is \;\; a \;\max imal \;\; element \;\; of \;\; P \\ \{Q \in Pl\; A \notin Q\} & if \;\; A \;\; is \;\; not \; a \; \max imal \;\; element \;\; of \;\; P \end{cases}$$

Observe that TA non-empty for every A ∈ P, and hence {TA}A∈P is a family of non-empty sets.

By the Axiom of Choice there is a family of sets {FA}A∈P such that FA ⊆ TA and FA has exactly one element for all A ∈ P. For each A ∈ P, let SA be the single element in FA.

By the definition of TA we see that SA ∈ P and A ⊆ SA for all A ∈ P; moreover, we have SA = A if and only if A is a maximal element of P.

To prove the theorem, it therefore suffices to find some M ∈ P such that SM = M.

Let R ⊆ P.

The family R is closed if A ∈ R implies SA ∈ R , and if C ⊆ R is a chain then

$$\bigcup_{C \in C} C \in R$$

By hypothesis the family P is closed. Let M be the intersection of all closed families in P.

We now prove four Points about M. Using these Points, we deduce the theorem, as follows.

Let $M = \bigcup_{C \in M} C \in R$ Point 4 says that M is a chain, and Point 1 says that M is closed.

It follows that M ∈ M. Again using the fact that M is closed, we deduce that SM ∈ M.

However, we know that C ⊆ M for all C ∈ M, and hence in particular that SM ⊆M.

As noted above, we know that M ⊆ SM, and we deduce that SM = M, and that is what needed to be proved.

Point 1. We will show that the family M is closed.

Let A ∈ M. Then A ∈ R for all closed families R ⊆ P, and

hence SA ∈ R for all closed families R ⊆ P, and

hence SA ∈ M. A similar argument shows that

if C ⊆ M is a chain then $\bigcup_{C \in C} C \in M$ the details are left to the reader.

Point 2. Let A ∈ M. Suppose that B ∈ M and B ≠ A imply SB ⊆ A. We will show that B ⊆ A or B ⊇ SA for all B ∈ M.

Let ZA = {C ∈ M | C ⊆ A or C ⊇ SA}.

We first show that ZA is closed.

First, let D ∈ ZA. Then D ∈ M, and D ⊆ A or D ⊇ SA.

Because M is closed, then SD ∈ M. Suppose first that D ⊆ A. If D ≠ A,

then by hypothesis on A we deduce that SD ⊆ A,

which implies that SD ∈ ZA.

If D = A, then SD = SA, and hence SD ⊇ SA,

which implies SD ∈ ZA.

Suppose second that D ⊇ SA. Because SD ⊇ D, it follows that SD ⊇ SA, which implies SD ∈ ZA.

Next, let C ⊆ ZA be a chain. Because M is closed, we know that $\bigcup_{C \in C} C \in M$

There are two cases. First, suppose that C ⊆ A for all C ∈ C.

Then it follows that $\bigcup_{C \in C} C \subseteq A$ and hence $\bigcup_{C \in C} C \subseteq Z_A$

Second, suppose that there is some E ∈ C such that E ∉ A. Because E ∈ ZA, then E ⊇ SA. Because $\bigcup_{C \in C} C \supseteq E$

it follows that

$\bigcup_{C \in C} C \supseteq S_A$ Hence $\bigcup_{C \in C} C \in Z_A$

We deduce that ZA is closed. Because M is the intersection of all closed families of sets in P, it follows that M ⊆ ZA.

On the other hand, by definition we know that ZA ⊆ M, and it follows that ZA = M. We deduce that B ⊆ A or B ⊇ SA for all B ∈ M.

Point 3. We will show that if A ∈ M, then B ∈ M and B ≠ A imply SB ⊆ A.

Let W = {A ∈ M | B ∈ M and B ≠ A imply SB ⊆ A}.

We first show that W is closed.

First, let F ∈ W. Then F ∈ M, and B ∈ M and B ≠ F imply SB ⊆ F. Because M is closed, we know that SF ∈ M.

Let G ∈ M, and suppose that G ≠ SF.

It follows that G ≠ SF. By Point 2 we know that G ⊆ F.

There are two cases.

First, suppose that G ≠ F. Then SG ⊆ F. Because F ⊆ SF, it follows that SG ⊆ SF.

Second, suppose that G = F. Then SG = SF, and hence SG ⊆ SF. We deduce that SF ∈ W.

Next, let C ⊆W be a chain. Because M is closed we know that $\bigcup_{C \in C} C \in M$

Let H ∈ M, and suppose that H $\bigcup_{C \in C} C$. If it were the case that C ⊆ H for all C ∈ C, then it would follow that $\bigcup_{C \in C} C \subseteq H$ which is not possible.

Hence there is some K ∈ C such that K⊄H.

Because K ∈W, then B ∈M and B⊄K imply SB ⊆ K.

By Point 2 we deduce that B ⊆ K or B ⊇ SK for all B ∈ M. Because SK ⊇ K, it follows that B ⊆ K or B ⊇ K for all B ∈ M.

Because K ≠ H, it follows that K ⊇ H. If K = H then it would follow that K ⊆ H, which is not true, and hence we deduce that H ≠ K.

It then follows that SH ⊆ K. Because K ∈ C, we deduce that

SH $\subseteq \bigcup_{C \in C} C$ Hence $\bigcup_{C \in C} C \in W$ We deduce that W is closed.

By an argument similar to the one used in **Point 2**, we deduce that W = M,

and

therefore we know that

if A ∈ M, then B ≠ A implies SB ⊆ A for all B ∈ M.

Point 4. We will show that M is a chain.

Let A,C ∈ M. By Point 3 we know that B ≠ A

implies SB ⊆ A for all B ∈ M,

and hence by Point 2 we deduce that B ⊆ A or B ⊇ SA for all B ∈ M.

Hence C ⊆ A or C ⊇ SA. Because SA ⊇ A,

it follows that C ⊆ A or C ⊇ A. We deduce that M is a chain.

Hence Proved.