Answer on Question #72278 – Math – Differential Geometry | Topology Question

Compute the torsion of the following curves

$(\boldsymbol{i})\ \ \gamma(t) = \left(\frac{4}{5}\cos t\,, 1 - \sin t\,, -\frac{3}{5}\cos t\right)$

$(\boldsymbol{ii})\ \ \gamma(t) = (t, \cosh t)$

$(\boldsymbol{iii})\ \ \gamma(t) = \frac{4}{5}(\cos^3 t\,, \sin^3 t)$

Solution

Torsion: $\tau(t) = \dfrac{(\overrightarrow{\gamma'}(t) \times \overrightarrow{\gamma''}(t)) \cdot \overrightarrow{\gamma'''}(t)}{\left\|\overrightarrow{\gamma'}(t) \times \overrightarrow{\gamma''}(t)\right\|^2}$

$(\boldsymbol{i})\ \ \gamma(t) = \left(\frac{4}{5}\cos t\,, 1 - \sin t\,, -\frac{3}{5}\cos t\right)$

We need to differentiate $\vec{\gamma}(t)$ three times. Thus

$$\overrightarrow{\gamma'}(t) = \left(-\frac{4}{5}\sin t\,, -\cos t\,, \frac{3}{5}\sin t\right)$$

$$\overrightarrow{\gamma''}(t) = \left(-\frac{4}{5}\cos t\,, \sin t\,, \frac{3}{5}\cos t\right)$$

$$\overrightarrow{\gamma'''}(t) = \left(\frac{4}{5}\sin t\,, \cos t\,, -\frac{3}{5}\sin t\right)$$

$$\overrightarrow{\gamma'}(t) \times \overrightarrow{\gamma''}(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\frac{4}{5}\sin t & -\cos t & \frac{3}{5}\sin t \\ -\frac{4}{5}\cos t & \sin t & \frac{3}{5}\cos t \end{vmatrix} = \vec{i}\left(-\frac{3}{5}\cos^2 t - \frac{3}{5}\sin^2 t\right) +$$

$$+\vec{j}\left(-\frac{12}{25}\sin t \cos t + \frac{12}{25}\sin t \cos t\right) + \vec{k}\left(-\frac{4}{5}\sin^2 t - \frac{4}{5}\cos^2 t\right) =$$

$$= \vec{i}\left(-\frac{3}{5}\right) + \vec{j}(0) + \vec{k}\left(-\frac{4}{5}\right)$$

$$\left(\overrightarrow{\gamma'}(t) \times \overrightarrow{\gamma''}(t)\right) \cdot \overrightarrow{\gamma'''}(t) = \left(-\frac{3}{5}\right)\left(\frac{4}{5}\sin t\right) + (0)(\cos t) + \left(-\frac{4}{5}\right)\left(-\frac{3}{5}\sin t\right) = 0$$

$$\left\|\overrightarrow{\gamma'}(t) \times \overrightarrow{\gamma''}(t)\right\|^2 = \left(\sqrt{\left(-\frac{3}{5}\right)^2 + (0)^2 + \left(-\frac{4}{5}\right)^2}\right)^2 = 1$$

$$\tau(t) = 0$$

If the torsion of a regular curve with non-vanishing curvature is identically zero, then this curve belongs to a fixed plane.

$(\boldsymbol{ii})\ \gamma(t) = (t, \cosh t)$

A plane curve with non-vanishing curvature has zero torsion at all points.

$\overrightarrow{\gamma'}(t) = (1, \sinh t\,, 0)$

$\overrightarrow{\gamma''}(t) = (0, \cosh t\,, 0)$

$\overrightarrow{\gamma'''}(t) = (0, \sinh t\,, 0)$

$$\overrightarrow{\gamma'}(t) \times \overrightarrow{\gamma''}(t) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & \sinh t & 0 \\ 0 & \cosh t & 0 \end{vmatrix} = \vec{k}(\cosh t)$$

$$\left(\overrightarrow{\gamma'}(t) \times \overrightarrow{\gamma''}(t)\right) \cdot \overrightarrow{\gamma'''}(t) = (0)(0) + (0)(\sinh t) + (\cosh t)(0) = 0$$

$$\left\|\overrightarrow{\gamma'}(t) \times \overrightarrow{\gamma''}(t)\right\|^2 = \left(\sqrt{(0)^2 + (0)^2 + (\cosh t)^2}\right)^2 = (\cosh t)^2 > 0$$

$\tau(t) = 0$

$(\boldsymbol{iii})\ \ \gamma(t) = \frac{4}{5}(\cos^3 t\,, \sin^3 t)$

A plane curve with non-vanishing curvature has zero torsion at all points.

$$\overrightarrow{\gamma'}(t) = \left(-\frac{12}{5}\sin t\cos^2 t\,, \frac{12}{5}\sin^2 t\cos t\,, 0\right)$$

$$\overrightarrow{\gamma''}(t) = \left(-\frac{12}{5}\cos^3 t + \frac{24}{5}\sin^2 t\cos t\,, -\frac{12}{5}\sin^3 t + \frac{24}{5}\sin t\cos^2 t\,, 0\right)$$

$$\overrightarrow{\gamma'''}(t) = \left(\frac{84}{5}\sin t\cos^2 t - \frac{24}{5}\sin^3 t\,, -\frac{84}{5}\sin^2 t\cos t + \frac{24}{5}\cos^3 t\,, 0\ \right)$$

$$\overrightarrow{\gamma'}(t) \times \overrightarrow{\gamma''}(t) =$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\frac{12}{5}\sin t\cos^2 t & \frac{12}{5}\sin^2 t\cos t & 0 \\ -\frac{12}{5}\cos^3 t + \frac{24}{5}\sin^2 t\cos t & -\frac{12}{5}\sin^3 t + \frac{24}{5}\sin t\cos^2 t & 0 \end{vmatrix} =$$

$$= -\frac{144}{25}\vec{k}(\sin^4 t\cos^2 t + \sin^2 t\cos^4 t) = -\frac{144}{25}(\sin^2 t\cos^2 t)\vec{k}$$

$$\left(\overrightarrow{\gamma'}(t) \times \overrightarrow{\gamma''}(t)\right) \cdot \overrightarrow{\gamma'''}(t) =$$

$$= (0)\left(\frac{84}{5}\sin t\cos^2 t - \frac{24}{5}\sin^3 t\right) + (0)\left(-\frac{84}{5}\sin^2 t\cos t + \frac{24}{5}\cos^3 t\right) +$$

$$+ \left(-\frac{144}{25}(\sin^2 t\cos^2 t)\right)(0) = 0$$

$$\left\|\overrightarrow{\gamma'}(t) \times \overrightarrow{\gamma''}(t)\right\|^2 = \left(\sqrt{(0)^2 + (0)^2 + \left(-\frac{144}{25}(\sin^2 t\cos^2 t)\right)^2}\right)^2 =$$

$$= \left(\frac{144}{25}\right)^2 \sin^4 t\cos^4 t$$

$$\tau(t) = 0$$